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Wavelet denoising techniques with applications to experimental geophysical data

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ABSTRACT

In this paper, we compare Fourier-based and wavelet-based denoising techniques applied to both synthetic and real experimental geophysical data. The Fourier-based technique used for comparison is the classical Wiener estimator, and the wavelet-based techniques tested include soft and hard wavelet thresholding and the empirical Bayes (EB) method. Both real and synthetic data sets were used to compare the Wiener estimator in the Fourier domain, soft thresholding, hard thresholding, and the EB wavelet-based estimators. Four synthetic data sets, originally designed by Donoho and Johnstone to isolate and mimic various features found in real signals, were corrupted with correlated Gaussian noise to test the various denoising methods. Quantitative comparison of the error between the true and estimated signal revealed that the wavelet-based methods outperformed the Wiener estimator in most cases. Also, the EB method outperformed the soft and hard thresholding methods in general because the wavelet representation is not sparse at the coarsest levels, which leads to poor estimation of the noise variance by the thresholding methods. Microseismic and streaming potential data from laboratory tests were used for comparison and showed similar trends as in the synthetic data analysis.

1. Introduction

Real world data rarely comes clean. In order to extract useful information from raw data, a theoretically sound and robust denoising method is often required. Here, we present and compare several nonparametric denoising techniques theoretically and numerically through synthetic and real experimental geophysical data, with the ultimate goal of helping one choose the appropriate techniques for different situations.

Before the development of a unifying theory for wavelets over the last 15 years, the classical Wiener filter operating in the Fourier domain was one of the most widely used denoising methods [1]. Although the Wiener filter operating in the Fourier domain is optimal among all linear estimators, it relies on the critical assumption that the real signal is circular stationary. This assumption is an idealization. Also, since the Wiener filter shrinks the Fourier coefficients of the data according to the signal-to-noise ratio (SNR), though the noise spectrum is in generally not known *a priori*, the SNR requires estimation. This can be difficult in the Fourier basis since the Fourier representation of the signal is often not sparse. The representation of a signal in a basis is sparse when the signal strength is concentrated in a few of the coefficients of the basis. The notion of sparsity of signal representation is fundamental in function estimation in the Gaussian noise model [2]. While the sparsity of representation of a signal is directly related to the choice of a basis, the Gaussian noise representation is not sparse in any given basis.

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Donoho and Johnstone made a fundamental breakthrough by showing that nonlinear thresholding estimators operating in the wavelet domain achieve nearly minimax risk over a large class of functions [3–5]. This property, together with their ease of implementation and generality, has led to the development of nonlinear thresholding estimators as a powerful function estimation tool. The crucial step in level-dependent wavelet thresholding is estimating the threshold at each level (scale) from the wavelet coefficients of the noisy data. Donoho and Johnstone showed that the optimal threshold at each level is proportional to the standard deviation of the noise. This can be estimated from the median of certain wavelet coefficients. The validity of the threshold estimation is based on the assumption that the wavelet representation of the deterministic signal is sparse, and thus most of the wavelet coefficients are due to noise. This assumption may be invalid when the signal does not have a sparse representation, especially at coarse levels, leading to error in estimating the optimal threshold at each level.

To fix this deficiency, several researchers have investigated an empirical Bayes (EB) approach for data denoising in a wavelet basis [6–12]. A mixture prior distribution is placed on the wavelet coefficients of the signal at each level, which is designed to capture the sparseness of the wavelet coefficients. The parameters in the prior and the noise variance are then estimated by maximizing the marginal log likelihood function from the noisy data. Subsequently, the signal can be estimated from the posterior distribution. Maximization in the EB approach usually requires iterative numerical methods, and therefore, the EB method can be slower than soft and hard thresholding techniques.

There are several parts to this paper. First, various important theoretical results are stated and discussed for the Wiener estimator, the soft and hard thresholding estimators, and the EB method. Second, the reasons that signal estimation in Gaussian noise is often better in a wavelet basis than in the Fourier basis are investigated. Third, it is shown how the EB method with a Gaussian mixture prior distribution on the signal wavelet coefficients avoids some of the deficiencies of soft and hard thresholding estimators. Finally, key properties of each estimator are studied and compared through both real and synthetic data examples.

The current work is applied in a experimental geophysical context: namely denoising signals from microseismic (MS) and streaming potential (SP) laboratory experiments. The estimated signals from the MS experiment can be used to determine p- and s-wave speeds of the test specimen and used for structural monitoring purposes, for example, fracture detection and localization. On the other hand, the estimated signals from the SP experiment can be used to determine the so-called SP coupling coefficient, which relates the voltage observed for a given fluid pressure change for flow in earth materials. The coupling coefficient estimated for different flow scenarios in the laboratory can then be used to predict the SP response in the field.

2. Theoretical results

2.1. Problem statement

The goal is to estimate the function $f(\cdot)$ from measured data $\{y_n\}$ where the model assumed is

$$y_n = f(t_n) + e_n, \quad n \in \mathscr{Z} \tag{1}$$

with $\{e_n\}$ stationary Gaussian noise.

2.2. Wiener filter

The classical Wiener estimator is the optimal linear estimator operating in the Fourier domain [1]. Let F, Y, and W be the discrete Fourier transform of signal f, data y, and noise e, and introduce an $N \times N$ matrix $G := (g_{ml})_{0 \leqslant m,l<N}$. Also, define $\tau^2 := \{\tau_m^2\}_{0\leqslant m<N}$ and $\sigma^2 := \{\sigma_m^2\}_{0\leqslant m<N}$ as the diagonal values of the covariance matrices of F and W. Assuming τ^2 and σ^2 are known, the Wiener estimate of the signal is

$$\hat{F}_m = \frac{\tau_m^2}{\tau_m^2 + \sigma_m^2} Y_m \tag{2}$$

with variance

$$V(\tau_m, \sigma_m) = \frac{\tau_m^2 \sigma_m^2}{\tau_m^2 + \sigma_m^2}. \tag{3}$$

The Wiener estimate of the signal $\hat{f}$ can now be obtained by the discrete inverse Fourier transform

$$\hat{f}_n = \sum_{m=0}^{N-1} h_{nm} \hat{F}_m, \tag{4}$$

where $\{h_{nm} := \exp(2\pi inm)/\sqrt{N}; n, m = 0, \ldots, N-1\}$ is the discrete Fourier basis. The Wiener estimator (2) shrinks the Fourier coefficients of the data based on the SNR towards 0; and the larger the noise power, the more the estimator shrinks the Fourier coefficients.

In practice, computing the Wiener estimator in (2) requires estimating the noise and signal variances (σ_m^2 and τ_m^2). This can be done by various techniques (see e.g. [13,14]). Once the noise and signal variance estimates ($\hat{\sigma}_m^2$ and $\hat{\tau}_m^2$) are obtained, an approximate standard deviation estimate s for the Wiener estimator can be estimated via (3) by

$$s = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} V(\hat{\tau}_m, \hat{\sigma}_m). \tag{5}$$

An approximate marginal 99% confidence interval for the Wiener estimator $\hat{f}$ can then be constructed as $[\hat{f}_n - 2.58s\hat{f}_n + 2.58s]$.

The Wiener estimator has the following problems: (1) the representation of real world signals that are spatially inhomogeneous is usually not sparse in the Fourier basis, so it is difficult to separate the noise from the signal, and (2) it is a linear estimator. On the other hand, these signals may have sparse representation in the wavelet basis, so it may be worthwhile to use a nonlinear estimator in the wavelet domain.