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Joint time–frequency offset detection using the fractional Fourier transform

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ABSTRACT

In this paper, a parametric correlation measure is introduced based on the fractional Fourier transform. Based on fundamental properties of this transform, maximum correlation is identified with a line structure in the time–frequency plane. Correlating measured and model data for different values of the parameter yields intersecting lines in the time–frequency plane, whose intersection is identified as the joint time–frequency offset of the measured data, as compared to the model data. Both theoretical and practical aspects of the method are discussed, while examples illustrate both the performance of the method and the geometric concept behind. The approach is particularly designed for analysing measured signals with different onset time (travel time) and frequency shift (Doppler), as compared to a given signal model.

1. Introduction

In physics, measured data are often used to resolve parameters in a physical model by comparing measured data with data to be expected from the model. For finding best matching parameters for such a physical model, correlating observed data and given ground truth data is a classical approach. In specific problems measured data differ from model data by means of onset/offset time and/or frequency modulation.

In satellite navigation for example [1,2], data are measured at a receiver with a time delay, depending on the distance from the satellite to the receiver. Furthermore, as the satellite is moving when transmitting signals a Doppler frequency shift is introduced. Knowing both time and frequency offsets is needed to decode the satellite's navigation message properly.

Also in other fields, like radar and sonar data processing joint detection of time delay and Doppler shift plays an important role. In these fields, an ambiguity function is commonly used to deal with this problem.

In Satellite Navigation literature, e.g. [1,2], correlating measured and model data is an appropriate way to detect both time- and frequency-offsets. However, correlation coefficients have to be computed for all possible pairs of time and frequency offsets. In this paper we introduce a correlator for detecting joint time–frequency offsets, based on the fractional Fourier transform (FRFT). Depending on a parameter α, the FRFT computes a representation of a signal along a line in the time–frequency domain through the origin and with tangent $\tan\alpha$. Varying α yields a set of equations, from which values for the joint offset can be computed. Redundancy in the set of equations is used to obtain a pool of solutions in the time–frequency plane. A unique solution for the time–frequency offset can be obtained from this, using statistics of such a pool.

The FRFT also appears in relation to the problem of joint time–frequency offset detection in radar and sonar, namely as part of the radon ambiguity transform (RAT) [3,4]. However, since this paper is devoted to a generalisation of classical (Fourier-based) correlation schemes, we will only briefly compare both FRFT-based methods.

The paper is organised as follows. In Section 2 we briefly discuss the FRFT and recall some of its properties to

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be used in the sequel of the paper. In Section 3 we introduce the idea of correlating using the FRFT. Furthermore, we discuss the aspect of redundancy when varying the parameter α. Section 4 deals with some practical remarks on the approach. We discuss possible ways of efficiently choosing values for α, a normalisation for the correlation measure and we consider frequency modulations in practice. Furthermore, the previously mentioned relation with the RAT is discussed. The approach is illustrated by means of two examples in Section 5. Also a geometric interpretation of the approach is treated in that section.

2. The FRFT

The FRFT was originally described by Kober [5] and was later introduced for signal processing by Namias [6]. This was done, starting from fractional powers of the eigenvalues of the Fourier transform and their corresponding eigenfunctions. With this formalism an integral representation of this operator was derived in a heuristic manner. In [7,8], McBride and Kerr provided a rigorous mathematical framework in which the formal work of Namias could be situated. We briefly discuss this mathematical framework, Namias' formal work and present properties of the transform, as are treated in more detail in [9–11].

The definition of the FRFT F_α of order α for signals $s \in L^2(\mathbb{R})$ is given by

$$F_\alpha[s](t) = \frac{C_\alpha}{\sqrt{2\pi|\sin\alpha|}} \int_{\mathbb{R}} s(u)\, \mathrm{e}^{\mathrm{i}\,((u^2+t^2)\cdot(\cot\alpha)/2 - ut\csc\alpha)}\, \mathrm{d}u, \quad (1)$$

for $0<|\alpha|<\pi$, with

$$C_\alpha = \mathrm{e}^{\mathrm{i}\,((\pi/4)\ \mathrm{sgn}\ \alpha - \alpha/2)}. \quad (2)$$

The parameter C_α in (2) guarantees that the energy of the signal is conserved through F_α.

For $\alpha = 0$ and $\alpha = \pi$ the FRFT is defined by

$$F_0[s](t) = s(t) \quad \text{and} \quad F_\pi[s](t) = s(-t). \quad (3)$$

For $\alpha \notin (-\pi, \pi]$ the FRFT is defined by periodicity $F_{\alpha+2\pi} = F_\alpha$. Particularly, we have from (1)

$$F_{\pi/2} = F \quad \text{and} \quad F_{n\pi/2} = F^n, \;\; n \in \mathbb{Z},$$

with F the Fourier transform for signals in $L^2(\mathbb{R})$.

A further relation with the classical Fourier transform on $L^2(\mathbb{R})$ is given by its eigenfunctions $h_k(t)$, the k-th order Hermite functions and corresponding eigenvalues $\mathrm{e}^{\mathrm{i}k\alpha}$. These eigenfunctions are the same as for the Fourier transform, while the eigenvalues coincide for $\alpha = \pi/2$. Based on the equation

$$F_\alpha h_k = \mathrm{e}^{\mathrm{i}k\alpha} h_k, \quad (4)$$

the first property of the FRFT we discuss here is derived straightforwardly

$$F_\alpha F_\beta s = F_{\alpha+\beta} s, \quad (5)$$

for all $\alpha, \beta \in \mathbb{R}$. This relation is commonly known as the index law.

Besides this property, for analysing signals in both time and frequency it is interesting to consider the relation of the FRFT with the time-shift operator and the frequency modulator, given by $T_{t_0}[s](t) = s(t - t_0)$ and $M_{\omega_0}[s](t) = \mathrm{e}^{\mathrm{i}\omega_0 t} s(t)$, respectively, for $t_0, \omega_0 \in \mathbb{R}$. For these operators, the following intertwining relations hold:

$$F_\alpha T_{t_0} = \mathrm{e}^{\mathrm{i}t_0^2(\sin 2\alpha)/4} M_{-t_0 \sin\alpha} T_{t_0\cos\alpha} F_\alpha, \quad (6)$$

$$F_\alpha M_{\omega_0} = \mathrm{e}^{-\mathrm{i}\omega_0^2(\sin 2\alpha)/4} M_{\omega_0\cos\alpha} T_{\omega_0\sin\alpha} F_\alpha, \quad (7)$$

see e.g. [11]. As a result of these relations we have

$$|F_\alpha M_{\omega_0} T_{t_0}[s](t)| = |F_\alpha[s](t - t_0\cos\alpha - \omega_0\sin\alpha)|, \quad (8)$$

which will be exploited by the α-modulus correlator in the next section.

A well-known property of the FRFT that also follows from (6) and (7) is its rotation property by means of the Wigner distribution. In fact, following [11] it can be shown that

$$W[F_\alpha s](t, \omega) = W[s](t\cos\alpha - \omega\sin\alpha, t\sin\alpha + \omega\cos\alpha), \quad (9)$$

with $W[s]$ the Wigner distribution. The same property also holds for the cross-Wigner distribution.

To finalise this brief introduction on the FRFT, we observe that for digitally processing a number of fast FRFT algorithm exists. For a discussion on various implementations we refer to [12].

3. The α-modulus correlator

A way to detect a time-offset t_0 in a measured signal $s(t)$ is to correlate the measured signal $T_{t_0}s$ with its groundtruth signal (if available) $s(t)$. The correlation $C_{s,T_{t_0}s}(t)$ given by

$$C_{s,T_{t_0}s}(t) = \int_{\mathbb{R}} s(u) T_{t_0} s(t+u)\, \mathrm{d}u \quad (10)$$

attains its maximum value $\|s\|^2$ at $t = t_0$, the time-offset. In other words

$$t_0 = \arg\max C_{s,T_{t_0}s}. \quad (11)$$

Detection of a frequency offset ω_0 can be realised in a similar way. Since

$$FM_{\omega_0}s(\omega) = Fs(\omega - \omega_0)$$

we have

$$\omega_0 = \arg\max C_{Fs,FM_{\omega_0}s}. \quad (12)$$

Identification of both time and frequency offsets in a signal can be achieved in a similar manner. However, then the problem becomes two-dimensional, since the maximum of (10) depends on both t_0 and ω_0. For sampled signals, this means that all possible offsets t_0 and ω_0 have to be considered, which can be very time consuming.

In the case of detecting time–frequency offsets in measured data, given a model of the signal, we can use a less robust but fast algorithm. Instead of comparing signals and their spectra we can also correlate the amplitude spectra of both the measured data and the given signal model. Since a translation only contributes to the phase of the signal, we have

$$\omega_0 = \arg\max C_{|Fs|,|FM_{\omega_0}T_{t_0}s|}. \quad (13)$$