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Fast communication

A fast algorithm for designing complementary sets of sequences

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ABSTRACT

In this paper, we introduce a fast computational frequency-domain approach for designing complementary sets of sequences. Following the basic idea of CAN-based algorithms, we propose an extension of the *CAN* algorithm to complement*ary* sets of sequences (which we call CANARY). Moreover, modified versions of the proposed algorithm are derived to tackle the complementary set design problems in which low peak-to-average-power ratio (PAR), unimodular or phase-quantized sequences are of interest. Several numerical examples are provided to show the performance of CANARY.

1. Introduction

An active sensing device such as a radar system, transmits suitable waveforms into its surrounding that enable it to measure useful properties (e.g. location or speed) of peripheral objects. The transmit waveforms are generally formulated by using discrete-time sequences (see e.g. [1]). Let $\boldsymbol{x}=(x(1),\ldots,x(N))^T$ represent such a sequence (to be designed). The aperiodic and, respectively, periodic autocorrelations of $\boldsymbol{x}$ are defined as

$$r(k)=\sum_{l=1}^{N-k} x(l)x^*(l+k)=r^*(-k), \quad 0\le k\le (N-1), \tag{1}$$

$$c(k)=\sum_{l=1}^{N} x(l)x^*(l+k)_{\mathrm{mod}\ N}=c^*(-k), \quad 0\le k\le (N-1). \tag{2}$$

In general, transmit sequences $\boldsymbol{x}$ with small out-of-phase (i.e. $k\neq 0$) autocorrelation lags lead to a better performance of an active sensing system. As a result, there exists a rich literature on designing such sequences (see e.g. [1–22] and the references therein).

In order to avoid non-linear side effects and maximize the efficiency of power consumption at the transmitter, unimodular sequences (with $|x(l)|=1$) are desirable. Moreover, for cases with more strict implementation demands, phase-quantized unimodular sequences must be considered. For unimodular sequences it is not possible to make all $\{|r(k)|\}$ much smaller than $r(0)$ (depending on the application, the needed ratio can be around 10^{-5} or even smaller). For instance, it can be easily observed that $|r(N-1)|=1$, no matter how we design the sequence $\boldsymbol{x}$. In contrast with this, unimodular sequences with zero out-of-phase (i.e. perfect) periodic autocorrelation can be obtained for example via construction algorithms [4]. However, even by considering the periodic correlation, finding phase-quantized unimodular sequences with perfect periodic autocorrelation is a hard task. The difficulties in designing sequences with good autocorrelation encouraged the researchers to consider the idea of complementary sets of sequences (CSS). A set $X=\{\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots,\boldsymbol{x}_M\}$

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containing M sequences of length N is called complementary iff the autocorrelations of $\{\boldsymbol{x}_m\}$ sum up to zero at any out-of-phase lag, i.e.

$$\sum_{m=1}^{M} r_m(k) = 0, \quad 1 \le |k| \le (N-1), \tag{3}$$

where $r_m(k)$ represents the kth autocorrelation lag of $\boldsymbol{x}_m$. Consequently, to measure the complementarity of a sequence set $\{\boldsymbol{x}_m\}$ one can consider the integrated sidelobe level (ISL) or the peak sidelobe level (PSL) metrics defined by

$$\mathrm{ISL} = \sum_{k=1}^{N-1} \left| \sum_{m=1}^{M} r_m(k) \right|^2,$$

$$\mathrm{PSL} = \max_k \left\{ \left| \sum_{m=1}^{M} r_m(k) \right| \right\}, \tag{4}$$

as well as the ISL-related merit factor (MF), i.e.

$$\mathrm{MF} = E^2/(2\mathrm{ISL}), \tag{5}$$

where E denotes the sum of the energy of the sequences. Complementary sets containing $M=2$ sequences, which are known as complementary pairs, form a special case of CSS. Complementary pairs with binary (i.e. ± 1) elements were first studied in [5] and are usually referred to as Golay pairs (GP).

CSS have been applied to radar pulse compression [7], multiple-input–multiple-output (MIMO) radars [8], ultrasonic ranging [9], synthetic aperture imaging [10], and ultrasonography [11]. In addition to active sensing systems, CSS have applications in code-division multiple-access (CDMA) communication schemes [12], ultra wideband (UWB) communications [13], orthogonal frequency-division multiplexing (OFDM) [14,15], channel estimation [16], and data hiding [17]. Due to such a wide range of applications, the construction of CSS has been an active area of research during the last decades. The majority of research results on CSS have been concerned with the analytical construction of GP or CSS for restricted sequence lengths N. For example, it is shown in [18] that GPs exist for lengths of the form $N = 2^{\alpha}10^{\beta}26^{\gamma}$ where α, β and γ are non-negative integers. Some conditions on the existence of CSS can be found in [19] and [20]. Furthermore, Ref. [20] considers the extension of GP to general CSS. A theoretical as well as computational investigation of feasible GPs of lengths $N < 100$ is accomplished in [21].

In contrast to analytical constructions, a computational design of CSS does not impose any restriction on the sequence length N or the set cardinality M. Furthermore, a computational algorithm for designing CSS can provide plenty of CSS without the need for user-tuned parameters of analytical constructions. Such algorithms can also be used to find almost (i.e. sub-optimal) complementary sets of sequences for (N,M) values for which no CSS exists. A computational algorithm (called ITROX) for designing CSS was introduced in [9]. In this paper, we propose an extension of the *CAN* algorithm [23] for designing complement*ary* sets of sequences (which we call CANARY). The proposed algorithm works in the frequency domain, and is generally faster than ITROX. This is due to the fact that ITROX is based on certain eigenvalue decompositions with $\mathcal{O}(MN^2)$ complexity, whereas CANARY relies on fast Fourier transform (FFT) operations with $\mathcal{O}(MN\log(N))$ complexity (the difference in computational burdens between the two algorithms can be clearly observed in practice when N grows large).

The rest of this work is organized as follows. Section 2 presents the CANARY algorithm for CSS design. The extension of the CANARY algorithm to phase-quantized (and other constrained) CSS is studied in Section 3. Section 4 is devoted to numerical examples, whereas Section 5 concludes the paper.

Notation: We use bold lowercase letters for vectors and bold uppercase letters for matrices. $(.)^T$, $(.)^*$ and $(.)^H$ denote the vector/matrix transpose, the complex conjugate, and the Hermitian transpose, respectively. $\mathbf{1}$ and $\mathbf{0}$ are the all-one and all-zero vectors/matrices, respectively. $\|\boldsymbol{x}\|_n$ or the l_n-norm of the vector $\boldsymbol{x}$ is defined as $(\sum_k |\boldsymbol{x}(k)|^n)^{1/n}$ where $\{\boldsymbol{x}(k)\}$ are the entries of $\boldsymbol{x}$. The Frobenius norm of a matrix $\boldsymbol{X}$ (denoted by $\|\boldsymbol{X}\|_F$) with entries $\{\boldsymbol{X}(k,l)\}$ is equal to $(\sum_{k,l} |\boldsymbol{X}(k,l)|^2)^{\frac{1}{2}}$, whereas the l_1-norm of $\boldsymbol{X}$ (denoted as $\|\boldsymbol{X}\|_1$) is given by $\sum_{k,l} |\boldsymbol{X}(k,l)|$. The matrix $e^{j\boldsymbol{X}}$ is defined element-wisely as $[e^{j\boldsymbol{X}}]_{k,l} = e^{j[\boldsymbol{X}]_{k,l}}$. arg(.) denotes the phase angle (in radians) of the vector/matrix argument. The symbol $\odot$ stands for the Hadamard (element-wise) product of matrices. $\mathbb{C}$ represents the set of complex numbers. Finally, δ_k is the Kronecker delta function which is equal to one when $k=0$ and to zero otherwise.

2. CANARY

It is well-known that for any sequence $\boldsymbol{x}$ of length N with aperiodic autocorrelation lags $\{r(k)\}$ (see e.g. [24])

$$\Phi(\omega) \triangleq \left| \sum_{n=1}^{N} x(n) e^{-j\omega n} \right|^2 = \sum_{k=-(N-1)}^{N-1} r(k) e^{-j\omega k} \tag{6}$$

where $\Phi(\omega)$ is the "spectrum" of $\boldsymbol{x}$. Consider a complementary set $X = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_M\}$ containing M sequences of length N. It follows from the Parseval equality that

$$2\mathrm{ISL} = \sum_{k=-(N-1)}^{N-1} \left| \sum_{m=1}^{M} r_m(k) - MN\delta_k \right|^2$$
$$= \frac{1}{2N} \sum_{p=1}^{2N} \left[\sum_{m=1}^{M} \Phi_m(\omega_p) - MN \right]^2 \tag{7}$$

with $\Phi_m(\omega_p)$ representing the spectrum of the mth sequence at the angular frequency $\omega_p = 2p\pi/(2N)$. Therefore, the minimization of the ISL metric in (4) can be accomplished by minimizing the following frequency-domain metric:

$$\sum_{p=1}^{2N} \left[\sum_{m=1}^{M} \left| \sum_{n=1}^{N} x_m(n) e^{-j\omega_p n} \right|^2 - MN \right]^2. \tag{8}$$

Inspired by the basic idea of the CAN algorithm in [23] that considers (8) with $M=1$, we propose a cyclic algorithm (which we call CANARY) for designing CSS.