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Image analysis by discrete orthogonal Racah moments

Hongqing Zhu[a,*], Huazhong Shu[a,c], Jun Liang[a], Limin Luo[a,c], Jean-Louis Coatrieux[b,c]

[a]*Department of Computer Science and Engineering, Laboratory of Image Science and Technology, Southeast University, 210096 Nanjing, People's Republic of China*
[b]*Laboratoire Traitement du Signal et de l'Image, Université de Rennes I – INSERM U642, 35042 Rennes, France*
[c]*Centre de Recherche en Information Biomédicale Sino-français (CRIBs)*

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Abstract

Discrete orthogonal moments are powerful tools for characterizing image shape features for applications in pattern recognition and image analysis. In this paper, a new set of discrete orthogonal moments is proposed, based on the discrete Racah polynomials. In order to ensure numerical stability, the Racah polynomials are normalized, thus creating a set of weighted orthonormal Racah polynomials, to define the so-called Racah moments. This new type of discrete orthogonal moments eliminates the need for numerical approximations. The paper also discusses the properties of Racah polynomials such as recurrence relations and permutability property that can be used to reduce the computational complexity in the calculation of Racah polynomials. Finally, we demonstrate Racah moments' feature representation capability by means of image reconstruction and compression. Comparison with other discrete orthogonal transforms is performed, and the results show that the Racah moments are potentially useful in the field of image analysis.

Keywords: Discrete orthogonal moments; Racah polynomials; Image reconstruction; Image compression

1. Introduction

Moments and functions of moments of image intensity values have been widely used in image processing and analysis such as invariant pattern recognition [1], image reconstruction [2], robust line fitting [3], edge detection [4], and image recognition [5]. Among the different types of moments, the Cartesian geometric moments are most widely used due essentially to their simplicity and their explicit geometric meaning. However, the geometric moments are not orthogonal, thus it is difficult to reconstruct the image from them. Teague [6] introduced the Legendre and Zernike moments using the corresponding orthogonal functions as kernels. It was proven that the orthogonal moments possess better image feature representation and are more robust to image noise compared to geometric moments [7]. Since both Legendre and Zernike moments are defined as continuous integrals over a domain of normalized coordinates, the computation of these continuous

*Corresponding author. Tel.: +86 25 83794249; fax: +86 25 83792698.
E-mail addresses: hqzhu@seu.edu.cn (H. Zhu), shu.list@seu.edu.cn (H. Shu).

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moments requires a suitable transformation of the image coordinate space and an appropriate approximation of the integrals [8,9], thus increasing the computational complexity and leading to discretization error. The discrete orthogonal polynomials have been recently introduced in the field of image analysis [10,11]. Mukundan et al. proposed a set of discrete orthogonal moment functions based on the discrete Tchebichef polynomials for image analysis tasks [10]. More recently, another new set of discrete orthogonal moment functions based on the discrete Krawtchouk polynomials was introduced in [11]. The use of discrete orthogonal polynomials as basis functions for image moments eliminates the need for numerical approximation, and satisfies exactly the orthogonality property in the discrete domain of image coordinate space. This property makes discrete moments superior to the conventional continuous orthogonal moments in terms of image representation capability.

As it is well known, the classical orthogonal polynomials of one discrete variable satisfy a difference equation of hypergeometric type. The discrete orthogonal polynomials can be classified into two categories [12]. The first one is the set of polynomials that are orthogonal on uniform lattice $\{x = 0, 1, 2, \ldots\}$. These orthogonal polynomials are solutions of the following difference equation [13]

$$\sigma(x)\Delta\nabla p_n(x) + \tau(x)\Delta p_n(x) + \lambda_n p_n(x) = 0, \tag{1}$$

where $\Delta p_n(x) = p_n(x+1) - p_n(x)$, $\nabla p_n(x) = p_n(x) - p_n(x-1)$ denote the forward and backward finite difference quotients, respectively. $\sigma(x)$ and $\tau(x)$ are functions of second and first degree, respectively, λ_n is an appropriate constant. The discrete Meixner, Krawtchouk, Charlier, Tchebichef, and Hahn polynomials belong to this category. The second one consists of the polynomials being orthogonal on non-uniform lattice $\{x = x(s), s = 0, 1, 2, \ldots\}$, which satisfy the following difference equation [13,14]

$$\tilde{\sigma}[x(s)]\frac{\Delta}{\Delta x(s-\frac{1}{2})}\left[\frac{\nabla y_n(s)}{\nabla x(s)}\right] + \frac{\tilde{\tau}[x(s)]}{2}\left[\frac{\Delta y_n(s)}{\Delta x(s)} + \frac{\nabla y_n(s)}{\nabla x(s)}\right] + \lambda_n y_n(s) = 0. \tag{2}$$

Here $\tilde{\sigma}(x)$ and $\tilde{\tau}(x)$ are polynomials in $x(s)$ of degree at most two and one, respectively, and λ_n is an appropriate constant. According to different non-uniform lattice functions, we may have different polynomials. For example, for non-uniform lattice $x(s) = s(s+1)$, we obtain the Racah or dual Hahn polynomials. For $x(s) = q^s$ or $(q^s - q^{-s})/2$, we have the q-Krawtchouk, or q-Meixner, or q-Charlier, or q-Hahn polynomials. For $x(s) = (q^s + q^{-s})/2$ or $(q^{is} - q^{-is})/2$, we have the q-Racah or q-dual Hahn polynomials [12,13].

In recent years, special attention has been paid to the study of the discrete orthogonal polynomials on non-uniform lattice [15–19]. Lattice field theories have become a powerful tool to avoid infinities in perturbative methods, and to obtain exact solutions of the field equations [20,21]. However, to the best of our knowledge, until now, no discrete orthogonal polynomial defined on non-uniform lattice has been used in the field of image analysis. In this paper, we address this problem by introducing a new set of discrete orthogonal polynomials, namely Racah polynomials, which are orthogonal on non-uniform lattice $x(s) = s(s+1)$. The Racah polynomials introduced by Askey and Wilson contain as limiting cases the classical polynomials of Jacobi, Laguerre and Hermite and their discrete analogues which go under the names of Hahn, Meixner, Krawtchouk and Charlier polynomials [22,23]. In physics, the Racah coefficients usually arise in atomic and nuclear shell model calculations. In modern mathematics, the Racah polynomials play a leading role in the theory of orthogonal polynomials of discrete variables and of finite difference equations [22].

The objective of this paper is to introduce the Racah polynomials into the field of image analysis and attempts to demonstrate their potential usefulness in this field. To achieve this, the Racah polynomials are first scaled to be within the range of $[-1, 1]$, so that the numerical stability of the polynomials can be assumed. The scaled Racah polynomials are then used as basis functions to define a new type of discrete orthogonal moments known as Racah moments. Similar to other discrete orthogonal moments, the error in the computed Racah moments due to discretization does not exist. Image can thus be exactly reconstructed from the complete set of discrete moments.

Although the Racah polynomials are orthogonal on non-uniform lattice, the discrete Racah moments defined in this paper apply to uniform pixel grid image. The difference between the Racah moments and the discrete moments based on the polynomials that are orthogonal on uniform lattice (e.g., discrete Tchebichef moments and Krawtchouk moments) is that the latter is directly defined on the image grid but, for the former, we should introduce an intermediate, non-uniform lattice, $x(s) = s(s+1)$.