Signal Processing 93 (2013) 500–510

Contents lists available at SciVerse ScienceDirect

Signal Processing

journal homepage: www.elsevier.com/locate/sigpro

# Gaussian filtering and smoothing for continuous-discrete dynamic systems

Simo Särkkä [a,*], Juha Sarmavuori [b]

[a] Department of Biomedical Engineering and Computational Science (BECS), Aalto University, Rakentajanaukio 2, 02150 Espoo, Finland
[b] Nokia Siemens Networks, Espoo, Finland

ARTICLE INFO

*Article history:*
Received 10 November 2011
Received in revised form
19 June 2012
Accepted 4 September 2012
Available online 11 September 2012

*Keywords:*
Bayesian continuous-discrete filtering
Bayesian continuous-discrete smoothing
Gaussian approximation
Kalman filter
Rauch–Tung–Striebel smoother

ABSTRACT

This paper is concerned with Bayesian optimal filtering and smoothing of non-linear continuous-discrete state space models, where the state dynamics are modeled with non-linear Itô-type stochastic differential equations, and measurements are obtained at discrete time instants from a non-linear measurement model with Gaussian noise. We first show how the recently developed sigma-point approximations as well as the multi-dimensional Gauss–Hermite quadrature and cubature approximations can be applied to classical continuous-discrete Gaussian filtering. We then derive two types of new Gaussian approximation based smoothers for continuous-discrete models and apply the numerical methods to the smoothers. We also show how the latter smoother can be efficiently implemented by including one additional cross-covariance differential equation to the filter prediction step. The performance of the methods is tested in a simulated application.



## 1. Introduction

Non-linear continuous-discrete optimal filtering and smoothing refer to applications of Bayesian inference to state estimation in dynamic systems, where the time behavior of the system is modeled as a non-linear stochastic differential equation (SDE), and noise-corrupted observations of the state are obtained from a non-linear measurement model. These kinds of continuous-discrete state estimation problems arise in many applications, such as, in guidance systems, integrated inertial navigation and passive sensor based target tracking [1–3]. Solving these estimation problems is very hard, because the SDEs appearing in the dynamic model or the corresponding Fokker–Planck–Kolmogorov partial differential equations cannot typically be solved analytically and approximations must be used. Here we consider the particularly difficult case of non-additive noise which is intractable to many existing methods in the field (cf. [4,5]).

In this paper, we show how the numerical integration based discrete-time Gaussian filtering and smoothing frameworks presented in [6–8] can be applied to continuous-discrete models. We first show how the recent numerical integration methods can be applied to the classical continuous-discrete Gaussian filtering framework [9]. The usage of cubature integration method in continuous-discrete filtering was also recently analyzed by Särkkä and Solin [5], and here we generalize those results. The main contributions of this paper are to derive new continuous-discrete Gaussian smoothers by using two different methods: (i) by forming a Gaussian approximation to the partial differential equations of the smoothing solution [10,11] and (ii) by computing the continuous limit of the discrete-time Gaussian smoother [8] as in [12]. Both of these smoothers consist of differential equations for the smoother mean and covariance. The third main contribution is to derive a novel computationally

* Corresponding author. Tel.: +358 50 512 4393;
fax: +358 9 470 23182.
*E-mail address:* simo.sarkka@aalto.fi (S. Särkkä).

0165-1684/$ - see front matter © 2012 Published by Elsevier B.V.
http://dx.doi.org/10.1016/j.sigpro.2012.09.002

efficient smoothing method which only requires forward integration of one additional matrix differential equation during the prediction step of the continuous-discrete filter.

### 1.1. Problem formulation

This paper is concerned with *Bayesian optimal filtering and smoothing* of non-linear continuous-discrete state space models [1] of the following form:

$$dx = f(x,t)\,dt + L(x,t)\,d\beta,$$
$$y_k = h_k(x(t_k),r_k), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state and $y_k \in \mathbb{R}^d$ is the measurement at time instant $t_k$. The functions $f(x,t)$ and $h_k(x,r)$ define the dynamic and measurement models, respectively. Here $\{\beta(t) : t \geq 0\}$ is a $s$-dimensional Brownian motion with diffusion matrix $Q(t)$ and $\{r_k : k = 1,2,\ldots\}$ is a Gaussian $\mathrm{N}(0,R_k)$ white random sequence. The processes $\beta(t)$ and $r_k$ as well as the random initial conditions $x(0) \sim \mathrm{N}(m_0,P_0)$ are assumed to be mutually independent. $L(x,t)$ is a matrix valued function which causes the effective diffusion matrix of the process noise to be

$$\Sigma(x,t) = L(x,t)Q(t)L^T(x,t), \tag{2}$$

and is thus allowed to be state-dependent. In this paper, we interpret the stochastic differential equations (SDE) as Itô-type stochastic differential equations (see, e.g., [13]).

In *continuous-discrete filtering*, the purpose is to compute the following *filtering distributions* which are defined for all $t \geq 0$, not only for the discrete measurement steps:

$$p(x(t)|\, y_1,\ldots,y_k), \quad t \in [t_k,t_{k+1}),\ k = 1,2,\ldots \tag{3}$$

The *Bayesian optimal continuous-discrete filter* [14,1,9] is actually almost the same as the discrete filter—only the prediction step is replaced with solving of the *Fokker–Planck–Kolmogorov (FPK) partial differential equation*.

In *continuous-discrete smoothing* we are interested in computing smoothing distributions of the form

$$p(x(t)|y_1,\ldots,y_K), \quad t \in [t_0,t_K]. \tag{4}$$

The formal Bayesian filtering and smoothing solutions to the state estimation problem – including the continuous-discrete special case – have already been around since the 1960s–1970s [15,1,10,11,16] and are in that sense well known. However, the only way to solve the formal Bayesian filtering and smoothing equations is by approximation, as the closed-form solution is available only for the linear Gaussian case [17–19] and for a few other isolated special cases (see, e.g., [20]). Although non-linear continuous-discrete optimal filtering and smoothing are mature subjects, the approximations have concentrated to Taylor series based methods [21–23,16,9] and other methods have received considerably less attention.

During the last few decades, the speed of computers has increased exponentially, and due to that, numerical integration methods and other computational methods have developed rapidly. Thus more accurate approximations to the formal filtering and smoothing equations are tractable than before. In particular, the sigma-point based unscented transform was introduced as an alternative to Taylor series approximations for discrete-time filters and estimators in [24–26] and the extension to smoothing problems was presented in [27]. The idea was extended to a full numerical integration based discrete-time Gaussian filtering and smoothing framework in [6–8]. Note that the Gaussian approximations themselves date back to the 1960s–1980s [28,1,9], but the contributions of the recent articles are in application of more general numerical integration methods to the problem.

Non-linear continuous and continuous-discrete smoothing has been more recently studied in [29,30], and applications of the unscented (sigma-point) transform and related approximations to continuous-discrete (and continuous-time) filtering and smoothing have been proposed in [30–32,12]. The extension of the cubature Kalman filter [33] to continuous-discrete filtering problems – using Itô–Taylor series based approximations – has also been recently studied in [4], and its relationship to the classical approach to continuous-discrete filtering was analyzed in [5].

In this paper we only consider Gaussian approximations, but obviously other approximations exist as well. One possible approach is to use MCMC (Markov chain Monte Carlo) based methods for sampling from the state posterior (see, e.g., [34–36]). It is also possible to use simulation (sequential Monte Carlo) based particle filtering and smoothing methods (see, e.g., [37,38]) or approximate the solution of the Fokker–Planck–Kolmogorov equation numerically (see, e.g., [39,40]). Although these methods are more accurate in some cases, they typically are computationally more demanding than Gaussian approximations.

## 2. Continuous-discrete gaussian filtering

### 2.1. Gaussian filter for the continuous-discrete problem

In the filtering algorithms presented in this paper, we use the classical Gaussian filtering [1,9,6,7] approach where the idea is to employ the approximation

$$p(x(t)|y_1,\ldots,y_k) \approx \mathrm{N}(x(t)|m(t),P(t)). \tag{5}$$

That is, we replace the true expectations with respect to $x(t)$ with expectations over the Gaussian approximation. The means $m(t)$ and covariances $P(t)$ are computed via the following algorithm:

1. *Prediction step*: Integrate the following mean and covariance differential equations starting from the mean $m(t_{k-1})$ and covariance $P(t_{k-1})$ on the previous update time, to the time $t_k$

$$\frac{dm}{dt} = \mathrm{E}[f(x,t)],$$
$$\frac{dP}{dt} = \mathrm{E}[(x-m)f^T(x,t)] + \mathrm{E}[f(x,t)(x-m)^T] + \mathrm{E}[\Sigma(x,t)], \tag{6}$$

where $\mathrm{E}[\cdot]$ denotes the expectation with respect to $x \sim \mathrm{N}(m,P)$. The results of the prediction are denoted as $m(t_k^-)$, $P(t_k^-)$, where the minus at superscript means 'infinitesimally before the time $t_k$'.