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Fast communication

Sparse signal representation by adaptive non-uniform B-spline dictionaries on a compact interval

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ABSTRACT

Non-uniform B-spline dictionaries on a compact interval are discussed in the context of sparse signal representation. For each given partition, dictionaries of B-spline functions for the corresponding spline space are built up by dividing the partition into subpartitions and joining together the bases for the concomitant subspaces. The resulting slightly redundant dictionaries are composed of B-spline functions of broader support than those corresponding to the B-spline basis for the identical space. Such dictionaries are meant to assist in the construction of adaptive sparse signal representation through a combination of stepwise optimal greedy techniques.

1. Introduction

A representation in the form of a linear superposition of elements of a vector space is said to be *sparse* if the number of elements in the superposition is small, in comparison to the dimension of the corresponding space. The interest for sparse representations has enormously increased the last few years, in large part due to their convenience for signal processing techniques and the results produced by the theory of Compressed Sensing with regard to the reconstruction of sparse signals from non-adaptive measurements [1–5]. Furthermore, the classical problem of expressing a signal as a linear superposition of elements taken from an orthogonal basis has been extended to consider the problem of expressing a signal as a linear superposition of elements, called *atoms*, taken from a redundant set, called a *dictionary* [6]. The corresponding signal approximation in terms of highly correlated atoms is said to be *highly nonlinear* and has been proved useful to signal processing applications. Moreover, a formal mathematical setting for highly nonlinear approximations is being developed. As a small sample of relevant literature let us mention [7–9].

In regard to sparse approximations there are two main problems to be looked at; one is in relation to the design of suitable algorithms for finding the sparse approximation, and other the construction of the dictionaries endowing the approximation with the property of sparsity. In this communication we consider the sparse representation matter for the large class of signals which are amenable to satisfactory approximation in spline spaces [10–12]. Given a signal, we have the double goal of (a) finding a spline space for approximating the signal and (b) constructing those dictionaries for the space which are capable of providing a sparse representation of such a signal by a combination of stepwise optimal greedy techniques. In order to achieve both aims we first discuss the construction of dictionaries of B-spline functions for non-uniform partitions, because the usual choice, the B-spline basis for the space, is not expected to yield sparse representations.

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In a previous publication [13] a prescription for constructing B-spline dictionaries on the compact interval is advised by restricting considerations to uniform partitions (cardinal spline spaces). Since our aim entails to relax this restriction, we are forced to look at the problem from a different perspective. Here we divide the partition into subpartitions and construct the dictionary by joining together the bases for the subspaces associated to each subpartition. Consequently, the uniform case considered in [13] arises as a particular case of this general construction. The capability of the proposed non-uniform dictionaries to produce sparse representations through a combination of stepwise optimal greedy techniques is illustrated by a number of examples.

The Communication is organized as follows: Section 2 introduces spline spaces and gives the necessary definitions. The property of spline spaces which provides us with the foundations for the construction of the proposed dictionaries is considered in this section (cf. Theorem 2). For a fixed partition, the actual constructions of non-uniform B-spline dictionaries is discussed in Section 3. Section 4 addresses the problem of finding the appropriate partition giving rise to the spline space suitable for approximating a given signal. In the same section a number of examples are presented, which illustrate an important feature of dictionaries for the adapted spaces. Through a stepwise optimal greedy selection of functions, they may render a very significant gain in the sparseness of the representation of those signals which are well approximated in the corresponding space. The conclusions are drawn in Section 5.

2. Background and notations

We refer to the fundamental books [14,15] for a complete treatment of splines. Here we simply introduce the adopted notation and the basic definitions which are needed for presenting our results.

Definition 1. Given a finite closed interval $[c,d]$ we define a *partition* of $[c,d]$ as the finite set of points

$$\Delta := \{x_i\}_{i=0}^{N+1}, N \in \mathbb{N}$$

$$\text{such that } c = x_0 < x_1 < \cdots < x_N < x_{N+1} = d. \tag{1}$$

We use Δ° to denote the point set $\{x_1,\ldots,x_N\}$, which is obtained by removing the boundary points from Δ. We further define N subintervals I_i, $i=0,\ldots,N$ as: $I_i=[x_i,x_{i+1})$, $i=0,\ldots,N-1$ and $I_N=[x_N,x_{N+1}]$.

Definition 2. Let Π_m be the space of polynomials of degree smaller or equal to $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let m be a positive integer and define

$$S_m(\Delta) = \{f \in C^{m-2}[c,d] : \ f|_{I_i} \in \Pi_{m-1}, i = 0,\ldots,N\}, \tag{2}$$

where $f|_{I_i}$ indicates the restriction of the function f to the interval I_i.

The standard result established by the next theorem is essential for our purpose.

Theorem 1 (*Schumaker [14, pp. 111]*). *Let*

$$\Delta := \{x_i\}_{i=0}^{N+1}, \ N \in \mathbb{N}$$

$$\text{such that } c = x_0 < x_1 < \cdots < x_N < x_{N+1} = d. \tag{3}$$

Then

$$S_m(\Delta) = \text{span}\{1, x, \ldots, x^{m-1}, (x-x_i)_+^{m-1}, i = 1, \ldots, N\},$$

where $(x-x_i)_+^{m-1} = (x-x_i)_{m-1}$ *for* $x-x_i > 0$ *and* 0 *otherwise.*

We are now ready to prove the theorem from which our proposal will naturally arise.

Theorem 2. *Suppose that* Δ_1 *and* Δ_2 *are two partitions of* $[c,d]$. *It holds true that*

$$S_m(\Delta_1) + S_m(\Delta_2) = S_m(\Delta_1 \cup \Delta_2).$$

Proof. It stems from Theorem 1 and the basic result of linear algebra establishing that for A_1 and A_2 two sets such that $S_1=\text{span}\{A_1\}$ and $S_2=\text{span}\{A_2\}$, one has $S_1+S_2 = \text{span}\{A_1 \cup A_2\}$.

Certainly, from Theorem 1 and for

$$A_1 := \{1, x, \ldots, x^{m-1}, (x-x_i)_+^{m-1}, x_i \in \Delta_1\} \quad \text{and}$$

$$A_2 := \{1, x, \ldots, x^{m-1}, (x-x_i)_+^{m-1}, x_i \in \Delta_2\}$$

we have: $S_m(\Delta_1) = \text{span}\{A_1\}$, $S_m(\Delta_2) = \text{span}\{A_2\}$. Hence

$$\begin{aligned} S_m(\Delta_1) + S_m(\Delta_2) &= \text{span}\{A_1 \cup A_2\} \\ &= \text{span}\{1, x, \ldots, x^{m-1}, (x-x_i)_+^{m-1}, x_i \in \Delta_1 \cup \Delta_2\} \end{aligned}$$

so that, using Theorem 1 on the right hand side, the proof is concluded. □

The next corollary is a direct consequence of the above theorem.

Corollary 1. *Suppose that* $\Delta_j, j = 1, \ldots, n$ *are partitions of* $[c,d]$. *Then*

$$S_m(\Delta_1) + \cdots + S_m(\Delta_n) = S_m\left(\bigcup_{j=1}^{n} \Delta_j\right).$$

3. Building B-spline dictionaries

Let us start by recalling that an *extended partition* with single inner knots associated with $S_m(\Delta)$ is a set $\tilde{\Delta} = \{y_i\}_{i=1}^{2m+N}$ such that

$$y_{m+i} = x_i, \quad i = 1, \ldots, N, \ x_1 < \cdots < x_N$$

and the first and last m points $y_1 \le \cdots \le y_m \le c$, $d \le y_{m+N+1} \le \cdots \le y_{2m+N}$ can be arbitrarily chosen.

With each fixed extended partition $\tilde{\Delta}$ there is associated a unique B-spline basis for $S_m(\Delta)$, that we denote as $\{B_{m,j}\}_{j=1}^{m+N}$. The B-spline $B_{m,j}$ can be defined by the recursive formulae [14]:

$$B_{1,j}(x) = \begin{cases} 1, & y_j \le x < y_{j+1}, \\ 0 & \text{otherwise,} \end{cases}$$

$$B_{m,j}(x) = \frac{x - y_j}{y_{j+m-1} - y_j} B_{m-1,j}(x) + \frac{y_{j+m} - x}{y_{j+m} - y_{j+1}} B_{m-1,j+1}(x).$$