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Accuracy analysis of the sine-wave parameters estimation by means of the windowed three-parameter sine-fit algorithm

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ABSTRACT

This paper investigates the accuracy of the sine-wave parameter estimators provided by the Weighted Three-Parameter Sine-Fit (W3PSF) algorithm when a generic cosine window is adopted. Simple expressions for the estimators are derived, which allows a very simple implementation of that algorithm. Moreover, it is shown that the W3PSF algorithm can be well approximated by the classical weighted Discrete Time Fourier Transform (DTFT) when the number of analyzed waveform cycles is high enough. Under that constraint the asymptotic mean square errors (MSEs) of the estimated parameters and the expected sum-squared fitting error can also be evaluated with good accuracy using very simple expressions. Then, the statistical performances of the W3PSF algorithm and the classical 3PSF algorithm are compared through computer simulations in the case of noisy or noisy and harmonically distorted sine-waves. The performed analysis allows us to identify when the W3PSF algorithm outperforms the classical 3PSF algorithm.

1. Introduction

Many engineering applications require accurate estimates of the parameters of noisy sine-waves. To this aim sine-fit algorithms based on the least squares approach are often adopted [1–15]. Indeed, these algorithms are quite simple to implement and return asymptotically efficient estimates when the analyzed sine-wave is affected by additive white Gaussian noise [1]. Two different sine-fit algorithms are available in the literature, that is the three-parameter sine-fit (3PSF) algorithm and the four-parameter sine-fit (4PSF) algorithm [3–6], according to the number of sine-wave parameters to be estimated. Both algorithms are recommended in the IEEE Standards for the dynamic characterization of digitizing waveform recorders [3] or analog-to-digital converters [4]. The 3PSF algorithm assumes that the sine-wave frequency is *a-priori* known, and the remaining waveform parameters are estimated through a linear least-square approach. Conversely, the 4PSF algorithm estimates also the sine-wave frequency, thus resulting in a non-linear least-square problem, whose solution is achieved iteratively. When the sine-wave frequency is known with uncertainty, a criterion for the selection of the most accurate algorithm has been derived in [7] on the basis of the parsimony principle [2]. The analytical expressions for the sine-wave parameter estimators provided by the 3PSF algorithm are given in [5]. Unfortunately, they are quite complicated unlike other approaches, such as those based on the Discrete Fourier Transform (DFT) [16–31]. In practice, sine-waves are often affected by narrow-band disturbances, such as harmonics or inter-harmonics. To reduce the detrimental effect of these disturbances the analyzed waveform is often weighted by means of a suitable window function in both non-parametric and parametric methods [1,2,16–18,20,22,27–29]. Unfortunately, to the best of the authors' knowledge, the accuracy of the weighted sine-fit algorithms has not yet been analyzed in the scientific literature. This is the aim of this paper. At first closed form expressions for the sine-wave parameter estimators returned by the Weighted 3PSF (W3PSF) algorithm are derived assuming that a generic H-term cosine window ($H \geq 1$) is adopted. The obtained expressions are simple linear combinations of the values of the weighted Discrete Time Fourier Transform (DTFT) evaluated at DC and at the sine-wave frequency. They allow us to analyze the effect of both narrow-band disturbances and wideband noise on the accuracy of the estimated sine-wave parameters. In particular, it is shown that the W3PSF algorithm and the classical weighted DTFT return almost the same estimates when quasi-coherent sampling occurs or when the number of analyzed sine-wave cycles is high enough. As a consequence, the expressions for the asymptotic (high number of samples) Mean Square Errors (MSEs) of the parameter estimates

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returned by the W3PSF algorithm and the related expected sum-squared fitting error [8] are easily achieved.

The paper is organized as follows. In Section 2 the expressions for the sine-wave parameter estimators returned by the W3PSF algorithm are derived and some remarks about the effect of narrow-band disturbances on the parameter estimation accuracy are drawn. Moreover, simple expressions for the asymptotic MSEs of the estimated parameters and the expected sum-squared fitting error provided by the W3PSF algorithm when the number of sine-wave cycles is high enough are provided in Section 3. In Section 4 the MSEs of the estimated parameters and the expected sum-squared fitting error of the W3PSF algorithm and the classical 3PSF algorithm are compared to each other in the case of noisy or noisy and harmonically distorted sine-waves. As a result, conditions under which the W3PSF algorithm outperforms the classical 3PSF algorithm are identified. Also, the MSEs returned by simulations are compared with the derived expressions. Finally, Section 5 concludes the paper.

2. The W3PSF algorithm

The analyzed discrete-time noisy sine-wave is modeled as:

$$x(m) = A\sin\left(2\pi f\left(m - \frac{M-1}{2}\right) + \phi\right) + d + e(m)$$
$$= s(m) + e(m), \quad m = 0, 1, 2, \ldots, M-1 \tag{1}$$

where $s(\cdot)$ is the discrete-time sine-wave of amplitude A, frequency f, and phase ϕ, d is the offset and $e(\cdot)$ is an additive white Gaussian noise with zero mean and variance σ^2. Without loss of generality an odd number of acquired samples M is assumed, so that the time reference can be set exactly in the center of the observation interval, thus simplifying the derived expressions (as it will be shown in the following).

The sine-wave frequency f is chosen in the range $(0, 0.5)$ to satisfy the Nyquist theorem. It can be expressed as:

$$f = \frac{f_{in}}{f_s} = \frac{\nu}{M} = \frac{l+\delta}{M}, \tag{2}$$

where f_{in} is the frequency of the acquired continuous-time sine-wave, f_s is the sampling frequency (both expressed in Hz), ν is the number of acquired signal cycles or the sine-wave frequency expressed in bins, l is the rounded value of ν, and $\delta = \nu - l$, with $\delta \in [-0.5, 0.5)$. In particular, $\delta = 0$ when coherent sampling occurs.

The sine-wave $s(\cdot)$ can be also re-written as:

$$s(m) = \frac{1}{2}\left(pe^{j2\pi f(m-\frac{M-1}{2})} + p^* e^{-j2\pi f(m-\frac{M-1}{2})}\right),$$
$$m = 0, 1, 2, \ldots, M-1 \tag{3}$$

where $p \triangleq Ae^{j\phi}$.

It is well-known that the influence on the estimated sine-wave parameters of narrow-band disturbances can be reduced by weighting the analyzed signal $x(\cdot)$ by a suitable window function. In this paper the cosine class windows ($H \geq 1$) are considered [32, 33], that is:

$$w(m) = \sum_{h=0}^{H-1} c_h \cos\left(2\pi\frac{h}{M}\left(m - \frac{M-1}{2}\right)\right), \quad m = 0, 1, \ldots, M-1 \tag{4}$$

where c_h, $h = 0, \ldots H-1$ are the window coefficients and $H \geq 1$ is the number of window terms. Note that the particular case $H = 1$ corresponds to the rectangular window, i.e. to the absence of the weighting.

The three-parameter sine-fit approach assumes that an estimate $\hat{\nu}$ of the number of acquired sine-wave cycles is known *a-priori* [3–7]. The following theorem holds.

Theorem 1. *The estimators for the sine-wave complex-amplitude p and the offset d returned by the W3PSF algorithm are:*

$$\hat{p} = \frac{2}{\tilde{W}(0)}\left(\alpha_{11}X_{\tilde{w}}(\hat{\nu}) + \alpha_{12}X_{\tilde{w}}^*(\hat{\nu}) + \alpha_{13}X_{\tilde{w}}(0)\right), \tag{5}$$

and

$$\hat{d} = \frac{1}{\tilde{W}(0)}\left(\alpha_{13}X_{\tilde{w}}(\hat{\nu}) + \alpha_{13}X_{\tilde{w}}^*(\hat{\nu}) + \alpha_{33}X_{\tilde{w}}(0)\right), \tag{6}$$

where $X_{\tilde{w}}(\cdot)$ is the Discrete Time Fourier Transform (DTFT) of the analyzed signal weighted by the squared window $w^2(m) = \tilde{w}(m)$, $\tilde{W}(\cdot)$ is the DTFT of the squared window $\tilde{w}(\cdot)$, and

$$\alpha_{11} \triangleq \frac{1-a^2}{1-2a^2-b^2+2a^2b}, \qquad \alpha_{12} \triangleq \frac{a^2-b}{1-2a^2-b^2+2a^2b},$$
$$\alpha_{13} \triangleq -\frac{a}{1+b-2a^2}, \qquad \alpha_{33} \triangleq \frac{1+b}{1+b-2a^2}, \tag{7}$$

in which

$$a \triangleq \frac{\tilde{W}(-\hat{\nu})}{\tilde{W}(0)} \quad \text{and} \quad b \triangleq \frac{\tilde{W}(-2\hat{\nu})}{\tilde{W}(0)}. \tag{8}$$

The proof of that theorem is given in Appendix A.

Since $p = Ae^{j\phi}$, the expressions for the sine-wave amplitude and phase estimators can be easily derived from (5):

$$\hat{A} = |\hat{p}| = \frac{2}{\tilde{W}(0)}\left|\alpha_{11}X_{\tilde{w}}(\hat{\nu}) + \alpha_{12}X_{\tilde{w}}^*(\hat{\nu}) + \alpha_{13}X_{\tilde{w}}(0)\right|, \tag{9}$$

and

$$\hat{\phi} = angle\{\hat{p}\} = angle\left\{\alpha_{11}X_{\tilde{w}}(\hat{\nu}) + \alpha_{12}X_{\tilde{w}}^*(\hat{\nu}) + \alpha_{13}X_{\tilde{w}}(0)\right\}. \tag{10}$$

It is worth noticing that the expressions (9), (10), and (6), for the estimators $\hat{A}$, $\hat{\phi}$, and $\hat{d}$ are explicitly related to the DTFT of the analyzed waveform weighted by the squared window $\tilde{w}(\cdot)$. Moreover, they are simpler than those published in the literature in the particular case of the rectangular window [5]. In addition, these expressions do not require the calculation of the pseudoinverse matrix, as occurs when using the classical least-squares approach.

Using mathematical induction it is easy to show the following

Proposition. *The squared window of a H-term cosine window ($H \geq 2$) with coefficients c_h, $h = 0, 1, \ldots, H-1$ is a $(2H-1)$-term cosine window with coefficients:*

$$\tilde{c}_0 = c_0^2 + 0.5\sum_{h=1}^{H-1} c_h^2 = NNPG_w, \tag{11a}$$

$$\tilde{c}_h = 2\sum_{n=-(H-1)}^{H-1-h} q_{|n|}q_{|n+h|}, \quad h = 1, 2, \ldots, 2H-2 \tag{11b}$$

in which $q_k = \begin{cases} c_0, & k=0 \\ 0.5c_k, & k=1,2,\ldots,H-1 \end{cases}$, while $NNPG_w$ is the Normalized Noise Power Gain parameter of the original window.

Other window figures of merit that will be used in the following, are the Normalized Peak Signal Gain (NPSG) and the Equivalent Noise BandWidth (ENBW). For the squared window $\tilde{w}(\cdot)$ the parameters NPSG, NNPG, and ENBW are expressed as follows [32]:

$$NPSG_{\tilde{w}} \triangleq \frac{1}{M}\sum_{m=0}^{M-1}\tilde{w}(m) = \tilde{c}_0 = NNPG_w, \tag{12}$$

$$NNPG_{\tilde{w}} \triangleq \frac{1}{M}\sum_{m=0}^{M-1}\tilde{w}^2(m) = \tilde{c}_0^2 + 0.5\sum_{h=1}^{H-1}\tilde{c}_h^2, \tag{13}$$