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Image denoising via sparse coding using eigenvectors of graph Laplacian

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ABSTRACT

Image denoising plays an important role in image processing, which aims to separate clean images from the noisy images. A number of methods have been presented to deal with this practical problem in the past decades. In this paper, a sparse coding algorithm using eigenvectors of the graph Laplacian (EGL-SC) is proposed for image denoising by considering the global structures of images. To exploit the geometry attributes of images, the eigenvectors of the graph Laplacian, which are derived from the graph of noised patches, are incorporated in the sparse model as a set of basis functions. Sequently, the corresponding sparse coding problem is presented and efficiently solved with a relaxed iterative method in the framework of the double sparsity model. Meanwhile, as the denoising performance of the EGL-SC significantly depends on the number of the used eigenvectors, an optimal strategy for the number selection is employed. A parameter called as out-of-control rate is set to record the percentage of the denoised patches that suffer from serious residual errors in the sparse coding procedure. Thus, with the eigenvector number increasing, the appropriate number can be heuristically selected when the out-of-control rate falls below an empirical threshold. Experiments illustrate that the EGL-SC can achieve a better performance than some other well-developed denoising methods, especially in the structural similarity index for the noise of large deviations.

1. Introduction

Image denoising plays an important role in image processing. It is viewed as an inverse problem in which clean images are estimated from the noisy images. A number of methods have been presented to deal with this practical problem in the past decades [1–4]. However, with the advances of sparse signal representation, more algorithms are efficiently proposed to resolve this problem [5–7]. It has already been proven that, compared with the traditional image denoising methods, these sparse representation algorithms are useful to improve image quality.

In brief, the basic idea of denoising methods via sparse representation can be described in two steps. First, image patches are approximately expressed by a linear combination of a few atoms taken from the dictionary. Second, clean images are separated from noise which is generally not compatible to the sparse assumption. Early denoising methods, e.g., K-means singular value decomposition (K-SVD) algorithm [8], usually treat the denoising problem as a pure approximation task, in which each patch is respectively estimated without consideration of the spatial redundancy to others. Therefore, these methods can be viewed as local methods for the neglect of the relationship among patches. Lately, numerous nonlocal denoising methods are developed to exploit such relationship. For example, one idea is to tie the sparse coefficients one to the other to deal with the deterioration of noisy images. Some constraints, i.e., grouping [9], clustering [10] and context-awareness [11], are used and very often applied in the sparse model as regularization terms. According to this idea, a so-called nonlocally centralized sparse representation method is recently presented, which centralizes sparse coefficients into various categories, and achieves the comparable performances with the block-matching and 3D filtering (BM3D) method [12–14].

On the other hand, though graph theory has been successfully employed in a variety of image applications, recent reports are further proven that it can be efficiently performed with sparse representation [15–17]. For example, various manifold regularized sparse coding methods, e.g., graph Laplacian [18], hypergraph Laplacian [19] and multi-modal sparse codings [20], are presented in image clustering, in which the graph Laplacian matrix is viewed as a useful tool to exploit the geometrical structures for image data [21–23]. As for image denoising via sparse representation, reports

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show that the denoising performance can be well improved by introducing the graph Laplacian. The corresponding sparse model is also proposed to regularize sparse coefficients via a manifold embedding term [24,25]. Generally, these graph-based denoising methods can be treated as non-local methods, in which the graph networks of pixels or patches are usually provided. Furthermore, very recently, a denoising method with the eigenvectors of the graph Laplacian (EGL) is developed to take advantage of the attributes of the eigenvectors [26]. In this method, the eigenvectors of the graph Laplacian of patches do not only represent the global structures of images, but also can be utilized as a set of basis functions to reconstruct images. Moreover, to enhance the denoising performance, only a certain part of the eigenvectors is used, where the number of these eigenvectors is experimentally set for each image and for the noise of each deviation, respectively. The corresponding experiments show the EGL can outperform several classical denoising methods including the K-SVD.

Motivated by recent progress, we propose a sparse coding algorithm using eigenvectors of the graph Laplacian (EGL-SC) for image denoising. The major contribution of our work relies in two aspects. First, the eigenvectors of the graph Laplacian of patches are employed in the sparse model to exploit the global structures of the noisy image. The eigenvector-based sparse coding problem is also presented, where a relaxed iterative solution is provided in the framework of the double sparsity model. Second, an optimal strategy is adopted to find the corresponding appropriate numbers of the eigenvectors adaptively for various images and for various noise deviations to further improve the denoising performance.

The rest of this paper is organized as follows. Some related work is reviewed in Section 2. The sparse coding algorithm using eigenvectors of the graph Laplacian is introduced in Section 3. Simulation results are presented in Section 4. Relevant conclusions and discussions are finally given in Section 5.

2. Related work

2.1. Sparse coding

In general, sparse representation can be classified in two tasks, i.e., dictionary learning and sparse coding (a.k.a. sparse approximation) [27,28]. The target of dictionary learning is to search an optimal signal space to support the attribution of a sparse vector under a certain measure. As for sparse coding, it is dedicated to find a sparse solution to an underdetermined linear system.

Given a measurement data matrix $\boldsymbol{Y} = [\boldsymbol{y}_1\ \boldsymbol{y}_2\ \dots\ \boldsymbol{y}_N]$, $\boldsymbol{y}_i \in \mathbb{R}^m$, the basic sparse coding problem can be described as two forms which are subject to the sparsity and residual error constraints respectively. The sparse coding problem under the residual error constraint is expressed as

$$\tilde{\boldsymbol{x}}_i = \arg\min_{\boldsymbol{x}_i} \|\boldsymbol{x}_i\|_0 \quad \text{s.t.} \quad \|\boldsymbol{y}_i - \boldsymbol{D}\boldsymbol{x}_i\|_2^2 \le \varepsilon, \quad i = 1, 2, \dots, N, \quad (1)$$

where $\boldsymbol{D} = [\boldsymbol{d}_1\ \boldsymbol{d}_2\ \dots\ \boldsymbol{d}_K]$ is the dictionary with the atoms $\{\boldsymbol{d}_i\}$, $\boldsymbol{d}_i \in \mathbb{R}^m$, $\boldsymbol{X} = [\boldsymbol{x}_1\ \boldsymbol{x}_2\ \dots\ \boldsymbol{x}_N]$ is the coefficient matrix with the sparse coefficients $\{\boldsymbol{x}_i\}$, $\boldsymbol{x}_i \in \mathbb{R}^K$, ε is a threshold of the residual error. In the image denoising application, each $\boldsymbol{y}_i$ represents a noisy image patch and N is the number of the total patches. Here, the number of the entries in $\boldsymbol{y}_i$ is set to be smaller than the number of the atoms, i.e., $m < K$. The dictionary is assumed to be a full-rank matrix. Thus, the solution of problem (1) is always available, where the residual error constraint only provides a searching area for $\boldsymbol{x}_i$.

Note that, problem (1) is formulated as an l_0-norm optimization problem, which is NP-hard. To deal with this problem, some greedy algorithms, e.g., matching pursuit [29] and orthogonal matching pursuit (OMP) [30], are successfully developed to achieve the approximated solution. However, in practice, this l_0-norm problem can be very often transformed into an l_1-norm problem, wherein the dictionary is in the control of the corresponding mutual incoherence [31]. Therefore, a number of effective algorithms, such as basis pursuit [32], iterative shrinkage [33] and split Bregman [34], are proposed to tackle the l_1-norm based sparse coding problems in different forms.

2.2. Denoising method with eigenvectors of the graph Laplacian

As for the EGL method, the main idea is to restore clean patches from noisy ones on a basis set of the eigenvectors of the graph Laplacian. More specifically, a patch-based undirect graph is firstly built on $\boldsymbol{Y}$ by using the k-nearest neighbors approach. Lately, a complex measure on the similarities and the location positions of patches is introduced as

$$d(i, j) = \|\boldsymbol{y}_i - \boldsymbol{y}_j\|_2 + \beta \left\| c(i) - c(j) \right\|_2, \quad (2)$$

where $d(i, j)$ is the complex measure for the patch pair $(\boldsymbol{y}_i, \boldsymbol{y}_j)$, β is a weighted coefficient and $c(i)$ represents the location position of $\boldsymbol{y}_i$. Thus, the weight matrix $\boldsymbol{W}$ can be constructed with its entries expressed as

$$w_{i,j} = \begin{cases} e^{-d^2(i,j)/\delta^2} & \text{if } \boldsymbol{y}_i \text{ is connected to } \boldsymbol{y}_j \\ 0 & \text{otherwise} \end{cases}, \quad (3)$$

where δ is a scaled coefficient. The normalized Laplacian matrix $\boldsymbol{L}$, $\boldsymbol{L} \in \mathbb{R}^{N\times N}$, is sequently obtained as

$$\boldsymbol{L} = \boldsymbol{I} - \boldsymbol{B}^{-1/2}\boldsymbol{W}\boldsymbol{B}^{-1/2}, \quad (4)$$

where $\boldsymbol{I}$ is an identity matrix, $\boldsymbol{B}$ is a diagonal matrix and its diagonal entries are the row sums of $\boldsymbol{W}$. Since the Laplacian matrix $\boldsymbol{L}$ is symmetric and positive semidefinite, its eigenvalues can be represented as $\{\lambda_i\}_{i=1}^N$, which are arranged in an ascending order with the first eigenvalue $\lambda_1 = 0$. The corresponding eigenvectors are given as $\{\boldsymbol{u}_i\}_{i=1}^N$, $\boldsymbol{u}_i \in \mathbb{R}^N$.

To deal with image denoising, the estimated measurement matrix $\tilde{\boldsymbol{Y}}$ is represented as

$$\tilde{\boldsymbol{Y}} = (\boldsymbol{Y}\boldsymbol{U})\boldsymbol{U}^T, \quad (5)$$

where $\boldsymbol{U} = [\boldsymbol{u}_1\ \boldsymbol{u}_2\ \dots\ \boldsymbol{u}_M]$ is a basis matrix with the first M eigenvectors. In (5), each row of $\boldsymbol{Y}$ is projected into a subspace spanned by the vectors of $\boldsymbol{U}$, and the estimation $\tilde{\boldsymbol{Y}}$ is reconstructed on such subspace.

In practice, the EGL employs an iterative procedure to deal with the noisy image, which is divided into two major stages. In the first stage, a rough image as a lowpass version of the clean image is estimated by using a very small number of the eigenvectors of the graph Laplacian from the noisy image. Lately, an intermediate image is constructed with a weighted average of the noisy and rough images. In the second stage, the denoised image is restored from the intermediate image by the corresponding eigenvectors. Here, the number of the eigenvectors used in second stage is heuristically set, which are fluctuated for various images and for noise of various deviations. In other words, to achieve a better denoising performance, the appropriate eigenvector number should be trivially tested for each image and for noise of each deviation, which may lead the EGL to be less effective.

3. Proposed algorithm

3.1. Sparse coding using eigenvectors of the graph Laplacian

We present the EGL-SC algorithm for image denoising. As an attempt to fully exploit the eigenvectors of the graph Laplacian, we modify the sparse coding problem (1) as