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Predictability on finite horizon for processes with exponential decrease of energy on higher frequencies

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A R T I C L E I N F O

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The paper presents sufficient conditions of predictability for continuous time processes in deterministic setting. We found that processes with exponential decay on energy for higher frequencies are predictable in some weak sense on some finite time horizon defined by the rate of decay. Moreover, this predictability can be achieved uniformly over classes of processes. Some explicit formulas for predictors are suggested.

1. Introduction

We study pathwise predictability of continuous time processes in deterministic setting and in the framework of the frequency analysis. It is well known that certain restrictions on frequency distribution can ensure additional opportunities for prediction and interpolation of the processes. The classical result is Nyquist–Shannon–Kotelnikov interpolation theorem for the band-limited processes. There are related predictability results; see, e.g., [13,1,2,11,6,10,9,12,7,8]. These works considered predictability of single processes, and the crucial assumption was that the processes are band-limited; the predictors were non-robust with respect to small noise in high frequencies; see, e.g., the discussion in Chapter 17 from [5].

We study some special weak predictability of continuous time processes. Instead of predictability of the original processes, we study predictability of sets of anticausal convolution integrals for a wide enough classes of kernels. This version of predictability was introduced in [3] for band-limited processes; it allowed to establish uniform predictability in this weakened sense over classes of band-limited and high-frequency processes. In the present paper, we established some predictability for continuous time processes with exponential decay of energy on the higher frequencies. It allows to consider processes that are not band-limited. More precisely, we obtain a sufficient condition of uniform weak predictability on prediction horizon T over some classes of processes with exponential decay of energy on higher frequencies $\omega\to\pm\infty$, when the energy is decreasing faster than $e^{-T|\omega|}$. An alternative formulation of this condition in time domain is also given. The predictors are obtained explicitly in the frequency domain via their transfer function. These predictors are defined entirely by the kernel of the convolution integral and their choice is independent from the characteristics of the particular input processes.

2. Problem setting and definitions

Let $x(t)$ be a currently observable continuous time process, $t\in\mathbf{R}$. The goal is to estimate, at a current time t, the values $y(t)=\int_t^{t+T}k(t-s)x(s)\,ds$, where $k(\cdot)$ is a given kernel, and $T>0$ is a given prediction horizon. At any time t, the predictors use historical values of the observable process $x(s)|_{s\le t}$.

We consider only linear predictors in the form $\widehat{y}(t)=\int_{-\infty}^{t}\widehat{k}(t-s)x(s)\,ds$, where $\widehat{k}(\cdot)$ is a kernel that has to be found. We will call $\widehat{k}$ a predictor or predicting kernel.

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Let us describe admissible classes of k and $\widehat{k}$.

Let $\mathbf{R}^+ \stackrel{\Delta}{=} [0,+\infty)$, $\mathbf{C}^+ \stackrel{\Delta}{=} \{z \in \mathbf{C} : \operatorname{Re} z > 0\}$, $i = \sqrt{-1}$.

For $x \in L_2(\mathbf{R}) \cup L_1(\mathbf{R})$, we denote by $X = \mathscr{F}x$ the function defined on $i\mathbf{R}$ as the Fourier transform of x;

$$X(i\omega) = (\mathscr{F}x)(i\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} x(t)\,dt, \quad \omega \in \mathbf{R}.$$

If $x \in L_2(\mathbf{R})$, then X is defined as an element of $L_2(\mathbf{R})$ (more precisely, $X(i\cdot) \in L_2(\mathbf{R})$).

For $x(\cdot) \in L_2(\mathbf{R})$ such that $x(t) = 0$ for $t < 0$, we denote by $\mathscr{L}x$ the Laplace transform

$$X(p) = (\mathscr{L}x)(p) \stackrel{\Delta}{=} \int_0^{\infty} e^{-pt} x(t)\,dt, \quad p \in \mathbf{C}^+. \tag{2.1}$$

Let H^r be the Hardy space of holomorphic on $\mathbf{C}^+$ functions $h(p)$ with finite norm $\|h\|_{H^r} = \sup_{s>0} \|h(s+i\omega)\|_{L_r(\mathbf{R})}$, $r \in [1,+\infty]$ (see, e.g., [4]).

Definition 1. For $T > 0$, we denote by $\mathscr{K}(T)$ the set of functions $k : \mathbf{R} \to \mathbf{R}$ such that $k(t) = 0$ for $t \notin [-T, 0]$ and such that $k \in L_\infty(\mathbf{R})$.

Definition 2. Let $\widehat{\mathscr{K}}$ be the class of functions $\widehat{k} : \mathbf{R} \to \mathbf{R}$ such that $\widehat{k}(t) = 0$ for $t < 0$ and such that $K(\cdot) = \mathscr{L}\widehat{k} \in H^2 \cap H^\infty$.

We consider below $k \in \mathscr{K}(T)$ and $\widehat{k} \in \widehat{\mathscr{K}}$.

Definition 3. Let $\bar{\mathscr{X}}$ be a class of processes $x(\cdot)$ from $L_2(\mathbf{R}) \cup L_1(\mathbf{R})$. Let $r \in [1,+\infty]$.

(i) We say that the class $\bar{\mathscr{X}}$ is L_r-predictable in the weak sense with the prediction horizon T if, for any $k(\cdot) \in \mathscr{K}(T)$, there exists a sequence $\{\widehat{k}_\gamma(\cdot)\}_{\gamma=1}^{+\infty} = \{\widehat{k}_\gamma(\cdot, \bar{\mathscr{X}}, k)\}_{\gamma=1}^{+\infty} \subset \widehat{\mathscr{K}}$ such that

$$\|y - \widehat{y}_\gamma\|_{L_r(\mathbf{R})} \to 0 \quad \text{as } \gamma \to +\infty, \ \forall x \in \mathscr{X},$$

where

$$y(t) \stackrel{\Delta}{=} \int_t^{t+T} k(t-s)x(s)\,ds, \quad \widehat{y}_\gamma(t) \stackrel{\Delta}{=} \int_{-\infty}^{t} \widehat{k}_\gamma(t-s)x(s)\,ds.$$

The process $\widehat{y}_\gamma(t)$ is the prediction of the process $y(t)$.

(ii) Let the set $\mathscr{F}(\bar{\mathscr{X}}) \stackrel{\Delta}{=} \{X = \mathscr{F}x, \quad x \in \bar{\mathscr{X}}\}$ be provided with a norm $\|\cdot\|$. We say that the class $\bar{\mathscr{X}}$ is L_r-predictable in the weak sense with the prediction horizon T uniformly with respect to the norm $\|\cdot\|$, if, for any $k(\cdot) \in \mathscr{K}(T)$, there exists a sequence $\{\widehat{k}_\gamma(\cdot)\} = \{\widehat{k}_\gamma(\cdot, \mathscr{X}, k, \|\cdot\|)\} \subset \widehat{\mathscr{K}}$ such that

$$\|y - \widehat{y}_\gamma\|_{L_r(\mathbf{R})} \to 0 \quad \text{uniformly in } \{x \in \bar{\mathscr{X}} : \|X\| \le 1\}.$$

Here $y(\cdot)$ and $\widehat{y}_\gamma(\cdot)$ are the same as above.

3. The main result

For $q \in \{1,2\}$, let $\mathscr{X}(q) = \mathscr{X}(q,T)$ be the set of processes $x(\cdot) \in L_2(\mathbf{R}) \cup L_1(\mathbf{R})$ such that

$$\int_{-\infty}^{+\infty} e^{qT|\omega|} |X(i\omega)|^q\,d\omega < +\infty, \quad X(i\omega) = \mathscr{F}x.$$

For $\Omega > 0$, set $D(\Omega) \stackrel{\Delta}{=} \mathbf{R} \backslash (-\Omega, \Omega)$.

Clearly, if $x(\cdot) \in \mathscr{X}(q,T)$, then

$$\int_{D(\Omega)} e^{qT|\omega|} |X(i\omega)|^q\,d\omega \to 0 \quad \text{as } \Omega \to +\infty.$$

It can be seen also that, for any $T > 0$, the class $\mathscr{X}(q,T)$ includes all band-limited processes x such that $X(i\omega) = \mathscr{F}x \in L_q(\mathbf{R})$, $q \in \{1,2\}$.

Theorem 1. *Let $q \in \{1,2\}$. Set $r = r(q) = +\infty$ if $q = 1$ and $r = r(q) = 2$ if $q = 2$.*

(i) *The class $\mathscr{X}(q,T)$ is L_r-predictable in the weak sense with the prediction horizon T.*

(ii) *Let $\mathscr{U}(q) = \mathscr{U}(q,T)$ be a class of processes $x(\cdot) \in \mathscr{X}(q,T)$ such that*

$$\int_{D(\Omega)} e^{qT|\omega|} |X(i\omega)|^q\,d\omega \to 0 \quad \text{as } \Omega \to +\infty$$

uniformly on $x(\cdot) \in \mathscr{U}(q)$.

Then this class $\mathscr{U}(q,T)$ is L_r-predictable in the weak sense with the prediction horizon T uniformly with respect to the norm $\|\cdot\|_{L_q(\mathbf{R})}$.

Some alternative descriptions and examples of sets $\mathscr{U}(q,T)$ are given below.

Remark 1. In [3], similar weak predictability with infinite horizon was introduced and established for models where an ideal low-pass filter exists; the predictors used in this paper were different from the ones presented below. Theorem 1 allows to extend this weak predictability on the case when the filters are not ideal but allow exponentially decay of energy on higher frequencies.

Remark 2. The case when processes $k(\cdot) \in L_2(\mathbf{R}) \backslash L_\infty(\mathbf{R})$ can also be covered. In this case, we have to require that $x \in L_2(\mathbf{R})$.

An example of a predictor: The question arises how to find the predicting kernels. We suggest a possible choice of the kernels; they are given explicitly in the frequency domain, i.e., via the transfer functions.

Let $k(\cdot) \in \mathscr{K}(T)$ and $K(i\omega) = \mathscr{F}k$. We assume here and below that $\omega \in \mathbf{R}$.

Let $\psi(\varepsilon) : (0,+\infty) \to \mathbf{R}$ be any function such that if $|z| \le \psi(\varepsilon)$ then $|e^z - 1| \le \varepsilon$, $z \in \mathbf{C}$. It follows from the continuity of the function e^z that $\psi(\varepsilon) > 0$ for any $\varepsilon > 0$. For example, one can select ψ as the inverse function to the modulus of continuity at zero for the exponent function e^z.

Lemma 1. *For $\gamma \in \mathbf{R}$, $\gamma > 0$, $p \in \mathbf{C}^+$, set*

$$g(p) \stackrel{\Delta}{=} T\frac{\gamma - p}{\gamma + p}p, \quad h(p) \stackrel{\Delta}{=} g(p) - Tp, \quad V(p) \stackrel{\Delta}{=} e^{h(p)}. \tag{3.1}$$

For $\omega \in \mathbf{R}$, set

$$\widehat{K}(i\omega) = V(i\omega)K(i\omega), \quad \widehat{k} = \mathscr{F}^{-1}\widehat{K}(i\omega). \tag{3.2}$$

Then $\widehat{K}(i\omega) \in L_2(\mathbf{R})$ and $\widehat{k}(\cdot) \in \widehat{K}(T)$. In addition, the predictability of the processes considered in Theorem 1 can be ensured with the sequence of the corresponding predicting kernels $\widehat{k} = \widehat{k}_\gamma$ defined by (3.1) *and* (3.2) *with $\gamma \to +\infty$. More*