Signal Processing 88 (2008) 2738–2746

Contents lists available at ScienceDirect

Signal Processing

journal homepage: www.elsevier.com/locate/sigpro

On discrete Gauss–Hermite functions and eigenvectors of the discrete Fourier transform

Balu Santhanam [a,*], Thalanayar S. Santhanam [b]

[a] Department of Electrical and Computer Engineering, MSC01 1100 1, University of New Mexico, Albuquerque, NM 87131-0001, USA
[b] Department of Physics, Saint Louis University, Missouri, MO 63103, USA

ARTICLE INFO

Article history:
Received 12 November 2007
Received in revised form
11 March 2008
Accepted 26 May 2008
Available online 20 June 2008

Keywords:
Discrete Fourier transform
Eigenvalues
Eigenvectors
Generalized K-symmetric matrices
Harmonic oscillator
Gauss–Hermite functions

ABSTRACT

The problem of furnishing an orthogonal basis of eigenvectors for the *discrete Fourier transform* (DFT) is fundamental to signal processing and also a key step in the recent development of discrete fractional Fourier transforms with projected applications in data multiplexing, compression, and hiding. Existing solutions toward furnishing this basis of DFT eigenvectors are based on the commuting matrix framework. However, none of the existing approaches are able to furnish a commuting matrix where both the eigenvalue spectrum and the eigenvectors are a close match to corresponding properties of the continuous differential Gauss–Hermite (G–H) operator. Furthermore, any linear combination of commuting matrices produced by existing approaches also commutes with the DFT, thereby bringing up issues of uniqueness.

In this paper, inspired by concepts from quantum mechanics in finite dimensions, we present an approach that furnishes a basis of orthogonal eigenvectors for both versions of the DFT. This approach furnishes a commuting matrix whose eigenvalue spectrum is a very close approximation to that of the G–H differential operator and in the process furnishes two generators of the group of matrices that commute with the DFT.

1. Introduction

Conventional Fourier analysis treats frequency and time as orthogonal variables and consequently is only suitable for the analysis of signals with stationary frequency content. The *fractional Fourier transform* (FRFT), an angular generalization of the Fourier transform, enables the analysis of waveforms, such as chirps, that possess time–frequency coupling. The continuous Fourier integral transform of a finite energy signal is defined via

$$X(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t) \exp(-\mathrm{j}\omega t)\,\mathrm{d}t = \mathscr{F}(x(t)).$$

Gauss–Hermite (G–H) functions defined by

$$H_n(t) = \frac{1}{\pi^{1/4}\sqrt{2^n n!}} h_n(t) \exp\left(-\frac{t^2}{2}\right),$$

where $h_n(t)$ is the nth-order Hermite polynomial, are solutions to the second-order differential equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} - (t^2 + \lambda)x(t) = 0.$$

They are eigenfunctions of the G–H differential operator

$$\mathscr{H}(x(t)) = (\mathscr{D}^2 - t^2 \mathscr{I})x(t) = -(2n+1)x(t),$$

with a corresponding eigenvalue of $\lambda_n = -(2n+1)$, where $\mathscr{D}, \mathscr{I}$ denote the derivative and identity operators. They are also eigenfunctions of the Fourier integral operator

$$\mathscr{F}(H_n(t)) = \exp\left(-\mathrm{j}n\frac{\pi}{2}\right)H_n(t),$$

* Corresponding author. Tel.: +1 505 277 1611; fax: +1 505 277 1439.
E-mail addresses: bsanthan@ece.unm.edu (B. Santhanam), santhats@slu.edu (T.S. Santhanam).

0165-1684/$ - see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.sigpro.2008.05.016

with a corresponding eigenvalue of $\lambda_n = \exp(-\mathrm{j}n\pi/2)$. These G–H functions are also eigenfunctions of the FRFT defined via

$$X_\alpha(u) = \int_{-\infty}^{\infty} x(t) K_\alpha(t,u)\,\mathrm{d}t,$$

$$K_\alpha(t,u) = \sum_{n=-\infty}^{\infty} \exp(-\mathrm{j}n\alpha) H_n(t) H_n(u).$$

Quantum mechanics as it pertains to the harmonic oscillator connects the canonical variables, position, and momentum through the Fourier integral operator $\mathscr{F}$ via [1,2]

$$\mathscr{F} = \exp\left(\mathrm{j}\frac{\pi}{4}(\hat{p}^2 + \hat{q}^2 - 1)\right),$$

where $\hat{q}$ and $\hat{p}$ are the position and momentum operators that are related through a similarity transformation [2]:

$$\hat{p} = \mathscr{F}\hat{q}\mathscr{F}^\dagger, \quad \hat{p} = -\mathrm{j}\frac{\mathrm{d}}{\mathrm{d}q},$$

where $\mathscr{F}^\dagger$ denotes the Hermitian adjoint of $\mathscr{F}$ and p, q denote the eigenvalues of their corresponding operators. In the continuous case the expression inside the exponential is exactly the G–H differential operator:

$$(\hat{q}^2 + \hat{p}^2)x(q) = -\frac{\mathrm{d}^2}{\mathrm{d}q^2}x(q) + q^2 x(q) = -\mathscr{H}(x(q)).$$

Consequently, G–H functions are also the eigenfunctions of the quantum harmonic oscillator. The position and momentum[1] operators furthermore do not commute and their commutator corresponds to the identity [2]

$$[\hat{q}, \hat{p}] = \hat{q}\hat{p} - \hat{p}\hat{q} = j\mathbf{I}. \tag{1}$$

The connection between the G–H operator and the Fourier transform $\mathscr{F}$ can be further expressed as

$$(\hat{p}^2 + \hat{q}^2 - 1)x(q) = -(\mathscr{H} + 1)(x(q)) = \frac{-4\mathrm{j}}{\pi}\log \mathscr{F}(x(q)). \tag{2}$$

This relation implies that in the continuous case the G–H operator is related to the logarithm of the Fourier transform.

2. Prior work

The eigenvalues and eigenvectors of the discrete Fourier transform (DFT) matrix have been of interest from early work [3], where the DFT eigenvalue problem was discussed in detail. Work in [4] outlines an analytical expression for the eigenvectors of the DFT corresponding to distinct eigenvalues. However, this expression involves infinite sums and is not computable. Recent efforts to develop a discrete version of the FRFT have focussed on the DFT and its centralized version and on generating an orthogonal basis of eigenvectors for the DFT by furnishing a commuting matrix that has a non-degenerate eigenvalue spectrum and shares a common basis of eigenvectors with the DFT. These approaches, however, do not yield a unique discretization since the sum or the product of matrices that commute with the DFT also commutes with the DFT. Our goal in this paper is to define a discrete equivalent of the G–H differential operator $\mathscr{H}$ that will furnish the basis for both the centered and off-centered versions of the DFT matrix. This framework will enable the definition of a discrete version of the FRFT and also serve as the discrete equivalent of the G–H operator with eigenvalues and eigenvectors that closely resemble those of the continuous counterpart.

Existing approaches toward obtaining an orthogonal basis of eigenvectors for the DFT can be grouped into two basic categories. The first approach called the **S** matrix approach or the Harper matrix approach [5–7] is based on replacing the derivatives in the G–H differential equation with finite differences thereby converting the differential equation into a difference equation. The approach furnishes an almost tridiagonal (Harper) matrix that commutes with the DFT matrix and consequently furnishes a basis of orthogonal DFT eigenvectors when N is not a multiple of four. Other numerical approaches that use orthogonal projections to furnish the eigenvectors of the Harper matrix **S** have been recently investigated in [8]. As shown in [5], the Harper matrix does not converge to the G–H operator in the limit, but rather to the Mathieu differential operator. Furthermore, the eigenvalue spectrum is not the linear spectrum with uniform spacing needed for consideration as the discrete G–H operator as described in Fig. 1.

The second approach pioneered by Grünbaum [9] and later refined in [10] is an algebraic approach that furnishes tridiagonal matrices that commute with both the centered and the off-centered versions of the DFT. It was shown in [9] that the commuting matrix in the limit converges to the G–H differential operator. However, the eigenvalues of the matrix do not exhibit the uniform integer spacing needed to be considered a viable candidate for the discrete G–H operator. Since the sum and the product of the different commuting matrices also commute with the DFT, numerous other commuting matrices can be furnished and the question of uniqueness of the commuting matrix approach arises. Recently, a combination of the commuting matrices from the Harper and Grünbaum matrix approaches have been used to furnish a basis of eigenvectors for the DFT [11,12], where the squared norm of error between the eigenvectors and the corresponding discrete G–H function was used to quantify the accuracy of the eigenvectors.

In this paper, we adopt a physical approach to develop a unique commuting matrix framework for both the CDFT and the DFT that: (a) furnishes a full orthogonal basis of eigenvectors resembling G–H functions via the eigenvalue problem for generalized K-symmetric matrices [13], (b) has an eigenvalue spectrum very close to that of $\mathscr{H}$, (c) converges to $\mathscr{H}$ in the limit, and (d) is quadratic in position and momentum analogous to the Hamiltonian of the quantum-mechanical harmonic oscillator.

3. Discrete G–H operator

3.1. Centered case

Toward formulating a physically meaningful, discrete, and computable version of the G–H operator, we borrow

[1] Although the analysis done here is done in terms of the quantum-mechanical variables $\hat{p}, \hat{q}$, they can be any pair of canonical variables such as time and frequency.