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On the behavior of Shannon's sampling series for bounded signals with applications ☆

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Abstract

In this paper we discuss the interplay between discrete-time and continuous-time signals and the question, whether certain properties of the signal in one domain carry over to the other domain. The Shannon sampling series and the more general Valiron interpolation series are the appropriate means to obtain the continuous-time, bandlimited signal out of its samples, i.e., the discrete-time signal. We study the effects of the projection operator, that projects the discrete-time signal on the positive time axis and, closely related, the differences in the convergence behavior of the symmetric and asymmetric Shannon sampling series. It is well known, that the space of discrete-time signals with finite energy and the space of continuous-time, bandlimited signals with finite energy are isomorphic. Thus, discrete-time and continuous-time, bandlimited signals with finite energy can be used interchangeably. This interchangeability is not restricted to finite energy signals. It is valid for a considerably larger class, but, as we show by stating an explicit example, not for the space of bounded signals: Even if the discrete-time signal is bounded, the corresponding bandlimited interpolation can be unbounded. In addition to the fact that the Shannon sampling series diverges for this signal, we prove that for this signal there is no bounded, bandlimited interpolation at all. Furthermore, by using the Banach–Steinhaus theorem we show that in a certain sense "almost all" signals have this property. Finally, some implications for practical applications are discussed.
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1. Introduction

In the classical paper "Communication in the Presence of Noise" [1,2], Shannon laid the fundamentals of communication and information theory. Other key contributions were made by Nyquist and Hartley [3,4]. An essential ingredient of [1] is the development of the sampling theorem, where the sampling series, which is called Shannon sampling series nowadays, plays a crucial role. Since then, this

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topic has become more and more popular and grown to an independent research area [5]. As a matter of fact sampling theory is the basis for all modern digital communications.

Sampling theory has its roots in the mathematical literature. Several mathematicians dealt with that topic: Borel, Hadamard, La Vallée Poussin and E.T. Whittaker are the most famous. The sampling series as it is known today was probably first described in 1915 by Whittaker in [6] where he called it cardinal function. However, Shannon [1] and Kotel'nikov [7] were the first to introduce the theory in the realm of communications. Thus, the Shannon sampling theorem is sometimes called Whittaker–Shannon–Kotel'nikov in order to recognize all of them. The reader who is further interested in the historical development of the sampling theorem is referred to [8–10], where several historical notes can be found.

In many fields of application the conversion of discrete-time to continuous-time signals and vice versa plays an important role. Therefore, it is an essential question, whether certain properties of the signal in one domain carry over to the other domain. It is a well known fact, that if the discrete signal has finite energy, then the bandlimited (continuous-time) signal, which is obtained by interpolation with Shannon's sampling series, has finite energy, too. Conversely, the sampled version of a bandlimited signal with finite energy has finite energy. By sampling and interpolation one can switch between both representations. Thus, the discrete-time signals with finite energy and the bandlimited continuous-time signals with finite energy can be used interchangeably. But, as we will show, for another even more important property, namely the boundedness of the signal, no interchangeability exists.

Shannon originally introduced the sampling theory for signals in L^2, but subsequently he used it—without a mathematical justification—for a much broader class of signals. The sampling theorem for signals with finite energy is examined under various aspects in [5], providing a good overview of the topic. In contrast, we examine the behavior of the sampling series for general bounded signals in this paper, since they are of particular importance. For example, bounded ± 1 sequences are the basis for BPSK symbols. The estimate of the peak value of the bounded bandlimited continuous-time signal, when the peak value of the samples is known, is an important problem. In the case of oversampling the peak-to-average power ratio (PAPR) is examined in [11] and used in [12] to find upper bounds on the statistical distribution of the crest-factor in orthogonal frequency-division multiplexing (OFDM) systems. In general, such an estimate is not possible if the samples are taken at the Nyquist rate.

The outline of this paper is as follows: In Section 2 we introduce some notations and in Section 3 we treat the concept of equivalent behavior of discrete-time and continuous-time signals. The main results are presented in Sections 4 and 5 and discussed in Section 6. Finally, we outline some consequences in Section 7.

2. Notation and preliminaries

As usual, $L^p(\mathbb{R})$, $1 \leqslant p < \infty$, denotes the space of all to the pth power Lebesgue integrable functions on $\mathbb{R}$, with the usual norm $\|\cdot\|_p$ and $L^\infty(\mathbb{R})$ the space of all functions for which the essential supremum norm $\|\cdot\|_\infty$ is finite. Furthermore, l^p, $1 \leqslant p < \infty$, is the space of all sequences such that the p-norm $\|\cdot\|_{l^p}$ is finite, and l^∞ denotes the space of bounded sequences with the supremum norm $\|\cdot\|_{l^\infty}$.

Let $\hat{f}$ denote the Fourier transform of a signal f, where $\hat{f}$ is to be understood in the distributional sense. For $\omega_g \geqslant 0$ let $\mathscr{B}_{\omega_g}$ be the set of all entire functions f with the property that for all $\varepsilon > 0$ there exists a constant $C(\varepsilon)$ with $|f(z)| \leqslant C(\varepsilon)\exp((\omega_g + \varepsilon)|z|)$ for all $z \in \mathbb{C}$. The Bernstein space $\mathscr{B}^p_{\omega_g}$ consists of all signals in $\mathscr{B}_{\omega_g}$, whose restriction to the real line is in $L^p(\mathbb{R})$, $1 \leqslant p \leqslant \infty$. A signal in is called bandlimited to ω_g.

The spectral support of signals in $\mathscr{B}_{\omega_g}$ can be characterized by the Paley–Wiener–Schwartz theorem [13–15], which is essentially an extension of the well-known Paley–Wiener theorem towards distributions. A version of the Paley–Wiener–Schwartz theorem can be found in [13]:

Theorem 1 (*Paley–Wiener–Schwartz*). *If $f \in \mathscr{B}_{\omega_g}$ is bounded by some polynomial on the real axis, then it has a Fourier transformation in the sense of distributions, and this Fourier transform is a distribution with compact support in $[-\omega_g, \omega_g]$. Conversely, if f is the inverse Fourier transform of a distribution with compact support in $[-\omega_g, \omega_g]$, then f belongs to $\mathscr{B}_{\omega_g}$ and is bounded by some polynomial on the real axis.*

By the Paley–Wiener–Schwartz theorem, the Fourier transform of a signal bandlimited to ω_g is supported in $[-\omega_g, \omega_g]$. It is well known, that $\mathscr{B}^p_{\omega_g} \subset$

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