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Signal Processing 87 (2007) 353–365
SIGNAL PROCESSING
www.elsevier.com/locate/sigpro

Prediction from off-grid samples using continuous normalized convolution☆

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Received 23 January 2003; received in revised form 23 June 2005; accepted 11 April 2006
Available online 12 June 2006

Abstract

This paper presents a novel method for performing fast estimation of data samples on a desired output grid from samples on an irregularly sampled grid. The output signal is estimated using integration of signals over a neighbourhood employing a local model of the signal using discrete filters. The strength of the method is demonstrated in motion compensation examples by comparing to traditional techniques.

Keywords: Irregular sampling; Filtering uncertain data; Reconstruction; Motion compensation; Resampling

1. Introduction

This paper presents a new method, continuous normalized convolution (CNC). The new method is an extension of normalized convolution (NC) [1] allowing for filtering signals represented by subpixel shifted samples. To illustrate the merits of CNC examples, this paper is focused on re-gridding irregularly sampled data, see Fig. 1. An efficient implementation is presented based on fast interpolation of basis filters to produce outputs on a desired sampling grid. The paper is organized as follows: Section 2 reviews the method NC [1]. In Section 3 we describe the new features of CNC. In Section 4 we consider filter design and implementational issues. Section 5 covers comparisons of CNC with more traditional methods and discusses its advantages using backward motion compensation as the driving example. Finally, we conclude the paper in Section 6.

2. Normalized convolution

In this section we will give a short description of the theory of NC, a method originally proposed by Knutsson and Westin [1,2]. NC is a general framework for filtering uncertain and sparsely sampled data. The method can be viewed as locally solving a weighted least square (WLS) problem, where the weights are defined by signal certainties and a

☆Supported by the Swedish Foundation for Strategic Research (SSF), the program Visual Information Technology (VISIT), and NIH Grant P41-RR13218.
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doi:10.1016/j.sigpro.2006.04.011

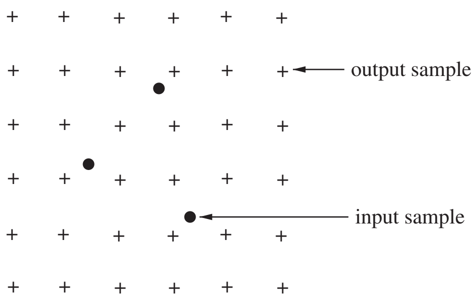

Fig. 1. Irregular sampling points (filled circle). Desired output sampling pattern (+).

spatially localizing weight function. Missing data in a sparse field is handled by setting the signal certainties to zero.

A key feature of NC is that solutions to the WLS systems can efficiently be calculated using a set of fixed convolution operators. Another important feature of NC is that the design of filters for irregular signals can be done using traditional methods and intuition gained for regularly sampled signals will still be valid.

Let $\mathbf{s}$ denote the neighborhood of a given signal. A local model of the signal can then be found using a weighted sum of basis functions $\mathbf{b}_k$. In our notation, the basis functions are column vectors, $\mathbf{b}_k$ in a matrix denoted $\mathbf{B}$:

$$\mathbf{B} = \begin{bmatrix} | & | & & | \\ \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_K \\ | & | & & | \end{bmatrix}. \tag{1}$$

The analysis is spatially localized by a positive scalar weighting function, $\mathbf{a}$, denoted the "applicability function". Similar to weighting the basis functions, the signal samples are weighted by a scalar certainty measure, c. Setting c to zero means that the value is unknown, which is equivalent to a missing sample.

NC solves the following WLS problem:

$$\tilde{\mathbf{r}} = \arg\min_{\mathbf{r}} \|\mathbf{W}_a\mathbf{W}_c(\mathbf{Br} - \mathbf{s})\|, \tag{2}$$

where $\mathbf{W}_a = \mathrm{diag}(\mathbf{a})$, $\mathbf{W}_c = \mathrm{diag}(\mathbf{c})$. With $\mathbf{W} = \mathbf{W}_a\mathbf{W}_c$, the solution is given by

$$\tilde{\mathbf{r}} = (\mathbf{B}^*\mathbf{W}^2\mathbf{B})^{-1}\mathbf{B}^*\mathbf{W}^2\mathbf{s}, \tag{3}$$

where $\mathbf{B}^*$ is the conjugate transpose of $\mathbf{B}$. Note that the coordinate vector $\mathbf{r}$ only exists if the matrix $(\mathbf{B}^*\mathbf{W}^2\mathbf{B})$ is invertible. This implies that there is a limit on the number of sample values that can be unknown. At least as many samples as the number of basis functions in $\mathbf{B}$ is required for the matrix to be non-singular, i.e. invertible. The signal $\mathbf{s}$ represented in the local basis $\mathbf{B}$ can be written as

$$\tilde{\mathbf{s}} = \mathbf{B}\tilde{\mathbf{r}}. \tag{4}$$

3. Continuous normalized convolution

Estimating signal values of data sampled at arbitrary real-valued positions requires continuous representations of the filter functions involved.

When estimating output values from an irregularly sampled input signal, each output sample is estimated from a neighborhood unique for that sample. CNC provides a computational efficient way to compute the values from a set of fixed regularly sampled filters using a linear model, i.e. a first-order spline. This alleviates the need to work with analytic filter functions that are recalculated for each neighborhood which greatly improves processing speed—as long as the number of operations for the filter approximation is less than the operations needed for an analytic description of the filter. Sometimes no analytic expression of the filter function exists that works in this approach.

This presentation will be focused on one- and two-dimensional (1D and 2D) signals. However, the extension to higher-dimensional signals is straightforward.

3.1. *Continuous filter approximation*

Although in principle not required, the continuous filter approximation in the following will be obtained using a regular grid, referred to as the filter grid. The results of the convolution of the irregularly sampled signal with the regularly sampled filter kernel will also be produced on a regularly sampled grid, referred to as the output grid. The filter coefficients corresponding to the irregularly sampled input points are produced using a first-order spline.

The first-order spline continuous filter approximation can be written as

$$\tilde{f}(x) = \bar{f} + \Delta x \cdot f_x, \tag{5}$$

where the filter coefficients are computed offline as

$$X = \begin{cases} \Delta \cdot (\lfloor \frac{x}{\Delta} \rfloor + 0.5) & \text{when } N \text{ is even}, \\ \Delta \cdot \mathrm{round}(\frac{x}{\Delta}) & \text{when } N \text{ is odd}, \end{cases} \tag{6}$$