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On the behavior of disk algebra bases with applications ☆

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Abstract

The present paper was motivated by an article [H. Akcay, On the existence of a disk algebra basis, Signal Processing 80 (2000) 903–907] on a basis in the disk algebra. Such bases play a central role for the representation of linear systems. In this article it was shown that the Lebesgue constant of a certain set of rational orthogonal functions in the disk algebra diverges. The present paper provides a generalization of this result. It shows that for any arbitrary complete orthonormal set of functions in the disk algebra the Lebesgue constant diverges. However, even if the Lebesgue constant diverges the orthonormal set may still be a disk algebra basis. Moreover, the paper discusses some implications of the divergence result with regard to the robustness of basis representations.

Keywords: Disk algebra; Orthonormal bases; Filter banks

1. Introduction

The decomposing of a linear system in terms of an orthonormal basis is a widely used tool in system and control theory, signal processing, communications or system identification and analysis. Assume that $f(e^{i\theta})$ with $\theta \in [-\pi, \pi)$ is the transfer function of a linear system $\mathscr{L}$ and let $\{\varphi_k(e^{i\theta})\}_{k=1}^{\infty}$ be a set of transfer functions of an orthonormal filterbank. Then, it is desirable to obtain an approximation of f in this filterbank of the form

$$(\mathscr{S}_N f)(e^{i\theta}) = \sum_{k=1}^{N} c_k(f) \cdot \varphi_k(e^{i\theta}) \quad (-\pi \leqslant \theta < \pi) \tag{1}$$

with constants $c_k(f)$ which are uniquely determined by f and such that $\mathscr{S}_N f$ converges to f as the degree N of the filterbank tends to infinity. The filterbank $\{\varphi_k\}_{k=1}^{\infty}$ will reflect the system theoretical properties of the linear systems $\mathscr{L}$, which should be approximated in this basis. This paper, for instance, considers causal and stable linear systems $\mathscr{L}$. Therefore, every single filterbank φ_k has to be causal and stable by itself. The first question is to know, does there exist such a filterbank such that all causal and stable linear systems $\mathscr{L}$ can be approximated in this filterbank, i.e. does the approximation (1) always converges to f as $N \to \infty$ for every

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arbitrary causal and stable transfer function f. Since the space of all causal and stable transfer functions is not a Hilbert space, the answer to this question is not that clear as in the Hilbert space case.

To make the discussion more precise, first the main notations are introduced in the following subsection. Afterward, Section 1.2 will give a more detailed problem formulation.

1.1. Notations and motivation

Throughout this paper $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ denotes the unit disk and the unit circle in the complex plane, respectively. Consistently, the following Banach spaces are used: $\mathscr{C}(\mathbb{T})$ is the space of continuous functions on $\mathbb{T}$. The norm in $\mathscr{C}(\mathbb{T})$ is the supremum norm $\|f\|_\infty = \sup_{\theta\in[-\pi,\pi)}|f(e^{i\theta})|$. As usual, $L^p = L^p(\mathbb{T})$ with $1 \leqslant p \leqslant \infty$ denotes the set of all p-integrable functions on $\mathbb{T}$. Especially important will be L^∞ equipped with the supremum norm as $\mathscr{C}(\mathbb{T})$, and the space L^2 which is a Hilbert space in the inner product

$$\langle f, g\rangle_2 := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\,d\theta \tag{2}$$

for all $f, g \in L^2$ and in which $\overline{g}$ denotes the conjugate complex of g. The norm in L^2 is given by $\|f\|_2 := \sqrt{\langle f, f\rangle_2}$. Every $f \in L^p$ with $p \geqslant 2$ can be represented by its *Fourier series*

$$f(e^{i\theta}) = \sum_{k=-\infty}^{\infty} \hat{f}_k e^{ik\theta} \quad \text{with}$$
$$\hat{f}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})e^{-ik\theta}\,d\theta. \tag{3}$$

The equality on the left-hand side of (3) means of course that the Fourier series converges in the L^p-norm to f, i.e. that $\lim_{N\to\infty}\|f(\cdot) - \sum_{k=-N}^{N}\hat{f}_k e^{ik(\cdot)}\|_p = 0$. The subspace of L^p of all functions $f \in L^p$ for which the *Fourier coefficients* $\hat{f}_k$ with negative index k are equal to zero ($\hat{f}_k = 0$ for all $k < 0$) are the so called *Hardy spaces* H^p. Every $f \in H^p$ can be identified with a function $f(z) := \sum_{k=0}^{\infty}\hat{f}_k z^k$ which is analytic inside the unit disk $\mathbb{D}$ and which is bounded with respect to the corresponding norm $\|f\|_p = \lim_{r\to 1}((1/2\pi)\int_{-\pi}^{\pi}|f(re^{i\theta})|^p\,d\theta)^{1/p}$. Finally, the *disk algebra* $\mathscr{A}(\mathbb{D})$ is the set of all functions $f \in H^\infty$ which are continuous in the closure $\overline{\mathbb{D}} = \mathbb{D} \cup \mathbb{T}$ of the unit disk.

Let $\mathscr{L}$ be a linear system with *impulse response* $\{\hat{f}_k\}_{k=-\infty}^{\infty}$ and *transfer function* $f(e^{i\theta})$, $\theta \in [-\pi, \pi)$ which are related by the Fourier transform (3). The linear system $\mathscr{L}$ is said to be *causal*, if $\hat{f}_k = 0$ for all $k < 0$, or equivalently if its transfer function $f(z)$ is an analytic function for all z inside the unit disk $\mathbb{D}$. Let $\{\hat{x}_n\}$ and $\{\hat{y}_n\} = \mathscr{L}[\{\hat{x}_n\}]$ be the input and output sequences of $\mathscr{L}$ with the corresponding Fourier transforms $x(e^{i\theta})$ and $y(e^{i\theta}) = \mathscr{L}[x(e^{i\theta})] = f(e^{i\theta}) \cdot x(e^{i\theta})$ with $\theta \in [-\pi, \pi)$, respectively. The L^2-norm of x and y can be interpreted as the average signal power. This energy norm of the signals induces a *stability norm* for the liner system $\mathscr{L}$ by

$$\|\mathscr{L}\|_S := \sup_{x\in L^2(\mathbb{T})} \frac{\|\mathscr{L}[x]\|_2}{\|x\|_2}.$$

It describes the maximal amplification of the signal power by $\mathscr{L}$. It can be shown that the stability norm of $\mathscr{L}$ is equal to the supremum norm of the transfer function f, i.e. $\|\mathscr{L}\|_S = \|f\|_\infty$. Thus, L^∞ can be identified with the set of all stable transfer function and H^∞ contains all causal and stable transfer function. However, since the space H^∞ is not separable, no basis will exists in H^∞. For this reason, the disk algebra $\mathscr{A}(\mathbb{D}) \subset H^\infty$ is considered. It is equal to the closure of all polynomials in H^∞, and therefore it is separable and a basis may exist in $\mathscr{A}(\mathbb{D})$.

This paper deals with causal and stable filter banks, i.e. complete sequences of orthonormal functions in H^2 which elements belong to the disk algebra. Thus, we consider sets of functions $\{\varphi_k\}_{k=1}^{\infty}$ with $\varphi_k \in \mathscr{A}(\mathbb{D})$ for all k, which are orthonormal with respect to the inner product (2), and for which the generalized Fourier series

$$(\mathscr{S}_N f)(z) = \sum_{k=1}^{N}\langle f, \varphi_k\rangle_2 \cdot \varphi_k(z) \tag{4}$$

converges to f for every $f \in H^2$, i.e. for which $\lim_{N\to\infty}\|f - \mathscr{S}_N f\|_2 = 0$. However, the generic convergence in H^2 does not imply the convergence in the stability norm $\|\cdot\|_\infty$ which is important for the stability behavior of the approximation $\mathscr{S}_N f$. Thus, given a causal and stable transfer function $f \in \mathscr{A}(\mathbb{D}) \subset H^2$, it is not clear at the outset whether or not the generalized Fourier series (4) converges to f in the stability norm. By the uniform boundedness principle (Theorem of Banach–Steinhaus [2]), a necessary and sufficient condition that $\lim_{N\to\infty}\|\mathscr{S}_N f - f\|_\infty = 0$ for all $f \in \mathscr{A}(\mathbb{D})$ is that the sequence of norms $\|\mathscr{S}_N\|_{\mathscr{A}(\mathbb{D})\to\mathscr{A}(\mathbb{D})}$ is uniformly bounded by a certain constant C_0. These operator