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# Robust controllability of interval fractional order linear time invariant systems ☆

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**Abstract**

We consider uncertain fractional-order linear time invariant (FO-LTI) systems with interval coefficients. Our focus is on the robust controllability issue for interval FO-LTI systems in state-space form. We revisit the controllability problem for the case when there is no interval uncertainty. It turns out that the controllability check for FO-LTI systems amounts to checking the controllability of conventional integer order state space. Based on this fact, we further show that, for interval FO-LTI systems, the key is to check the linear dependency of a set of interval vectors. Illustrative examples are presented.


*Keywords:* Fractional order systems; Robust controllability; Interval linear time invariant systems; Interval matrix; Linear dependency of interval vectors

## 1. Introduction

Based on fractional order calculus [1–4], fractional order dynamic systems and controls have been gaining increasing attention in research communities [5–9]. Pioneering works in applying fractional calculus in dynamic systems and controls include [10–13] while some recent developments can be found in [14–16].

Stability and controllability concepts are fundamental to any dynamic control systems including fractional order control systems [17,18]. In [19–24], stability results of fractional order control systems were presented while in [25], the first discussion about the controllability of fractional order control systems can be found. For interval FO-LTI systems, the first result on stability was discussed in [26] and further in [27] with even interval uncertainties (in the fractional order!). However, the controllability issue for interval FO-LTI systems has never been addressed. In this paper, we will present a method for checking the robust controllability for FO-LTI systems in the state space form. Based on the results of [28,29], we address the robust controllability issue via a sufficient linear independency condition of interval vectors. Note that, nobody has presented

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any property about the interval vectors except [28] although the interval vector concepts have been introduced in [30,31]. Furthermore, in robust control, the model uncertainty has been effectively and popularly handled by "interval" concept. Great amount of literatures are available under the term "interval" such as interval algebra [30,31], interval polynomial [32,33], Schur stability of interval matrices [34,35], Hurwitz stability of interval matrices [36–38], interval polynomial matrices [39], eigenvalues of interval matrices [40–42], and robust control with parameter uncertainty [43,44]. It is obviously beneficial to consider interval fractional order system as in [27,26]. For the ease of our presentation, we first re-visit the controllability issue of FO-LTI mainly based on [25]. Then, we briefly present the robust controllability issue of interval FO-LTI systems based on the concept of linear dependency of inter vectors [28]. Some examples will be given for illustrations.

## 2. Controllability of FO-LTI systems revisited

We adopt the Caputo definition for fractional derivative of order $\alpha$ of any function $f(t)$ [45,46]:

$$\frac{\mathrm{d}^{\alpha}f(t)}{\mathrm{d}t^{\alpha}}=\frac{1}{\Gamma(\alpha-n)}\int_0^t\frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}\,\mathrm{d}\tau,\quad (n-1<\alpha\leqslant n). \tag{1}$$

Based on the definition of (1), the Laplace transform of the fractional derivative is

$$\mathscr{L}\left\{\frac{\mathrm{d}^{\alpha}f(t)}{\mathrm{d}t^{\alpha}}\right\}=s^{\alpha}F(s)-\sum_{k=0}^{n-1}f^{k}(0^{+})s^{\alpha-1-k}. \tag{2}$$

In general, an LTI FOS can be described by the differential equation or the corresponding transfer function of non-commensurate real orders of the following form:

$$G(s)=\frac{b_m s^{\beta_m}+\cdots+b_1 s^{\beta_1}+b_0 s^{\beta_0}}{a_n s^{\alpha_n}+\cdots+a_1 s^{\alpha_1}+a_0 s^{\alpha_0}}=\frac{Q(s^{\beta_k})}{P(s^{\alpha_k})}, \tag{3}$$

where $\alpha_k,\beta_k$ $(k=0,1,2,\ldots)$ are real numbers and without loss of generality they can be arranged as $\alpha_n>\cdots>\alpha_1>\alpha_0$, $\beta_m>\cdots>\beta_1>\beta_0$.

In the particular case of *commensurate order* systems, it holds that, $\alpha_k=\alpha k,\beta_k=\alpha k,(0<\alpha<1)$, $\forall k\in \mathrm{Z}$, and the transfer function has the following form:

$$G(s)=K_0\frac{\sum_{k=0}^{M}b_k(s^{\alpha})^k}{\sum_{k=0}^{N}a_k(s^{\alpha})^k}=K_0\frac{Q(s^{\alpha})}{P(s^{\alpha})}. \tag{4}$$

With $N>M$, the function $G(s)$ becomes a proper rational function in the complex variable $s^{\alpha}$ which can be expanded in partial fractions of the following form:

$$G(s)=K_0\left[\sum_{i=1}^{N}\frac{A_i}{s^{\alpha}+\lambda_i}\right], \tag{5}$$

where $\lambda_i\,(i=1,2,\ldots,N)$ are the roots of the polynomial $P(s^{\alpha})$ or the system poles which are assumed to be simple without loss of generality. Then, it is straightforward to consider the following fractional order LTI system in state-space form

$$\frac{\mathrm{d}^{\alpha}x(t)}{\mathrm{d}t^{\alpha}}=Ax(t)+Bu(t), \tag{6}$$

where $\alpha\in(0,1]$, $x\in\mathscr{R}^n$, $u\in\mathscr{R}^r$, $A\in\mathscr{R}^{n\times n}$, $B\in\mathscr{R}^{n\times r}$, $\mathrm{rank}(B)=r$.

Similar to the conventional controllability concept [17], the controllability of (6) is defined as follows:

**Definition 2.1.** The FO-LTI system (6) is said to be controllable on $[t_0,t_f]$ iff for any initial state $x(t_0)$ and final state $x(t_f)$, there exists a control function $u(t)$ defined on $[t_0,t_f]$ which can drive the initial state $x(t_0)$ to the final state $x(t_f)$.

In what follows, we will show that, the controllability condition is the same as the integer order case. First, the solution of (6) is given by

$$X(s)=(s^{\alpha}I-A)^{-1}s^{\alpha-1}x(t_0)+(s^{\alpha}I-A)^{-1}BU(s) \tag{7}$$

in Laplace $s$-domain and

$$x(t)=E_{\alpha,1}(At^{\alpha})x(t_0)+\int_{t_0}^{t}(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(A(t-\tau)^{\alpha})Bu(\tau)\,\mathrm{d}\tau \tag{8}$$

in time-domain where $E_{\alpha,\beta}(z)$ is the Mittag–Leffler function in two parameters defined as

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}\quad(\alpha>0,\beta>0), \tag{9}$$

a generalization of exponential function, i.e.,

$$E_{1,1}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+1)}=\sum_{k=0}^{\infty}\frac{z^k}{k!}=e^z.$$

Now, for any given $t_0$, $t_f$ and the states $x(t_0)$ and $x(t_f)$, let us see under what condition there exists a unique control function $u(t)$ for $t\in[t_0,t_f]$. From (8),