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New reconstruction formulas for oversampled processes and functions

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Abstract

This paper addresses the reconstruction of band-limited oversampled stationary processes and functions. The reconstruction is performed from a multiperiodic subset of the periodic sampling sequence and from some isolated samples. Reconstruction performance can be characterized at the omitted sample points. The omission of some sample points provides a time-varying nature to the reconstruction formulas. This particular sampling scheme associated to specific interpolation functions result in an exact reconstruction with an arbitrarily tunable convergence rate. Moreover, the convergence properties hold when the reconstruction is performed in the neighbourhood of any lost sample. Indeed, the formulas can be fitted to any sample loss or deterioration by a simple time index translation. Specific expressions of the general reconstruction formula are derived for different process bandwidth ranges.

Keywords: Multiperiodic sampling; Linear interpolation; Oversampling; Stationary processes; Band-limited functions

1. Introduction

The theory of sampling and interpolation of functions or processes was initiated before the 19th century [1]. Developments of functions in terms of their values at integer points have been studied in various mathematical frameworks. For instance, Cauchy [2] has provided such developments in the framework of complex variable theory. This paper deals with the reconstruction of continuous-time band-limited stationary processes and functions from part of their periodically sampled observations. The theory and examples are mainly presented for random processes. However, since the random case implies more complex mathematical developments, the transposition to deterministic functions is straightforward. Simulations have been performed in both cases.

In what follows, $\mathbf{Z} = \{Z(t), t \in \mathbb{R}\}$ is a real or complex stationary zero-mean process with power spectral density $s(\omega)$ defined by

$$E[Z(t)Z^*(t-\tau)] = \int_{-\pi+a}^{\pi-a} e^{i\omega\tau} s(\omega)\, d\omega, \quad 0 < a < \pi. \tag{1}$$

$E[..]$ stands for mathematical expectation and * for complex conjugate. Parameter a is related to $\mathbf{Z}$ oversampling: the bandwidth of $\mathbf{Z}$ is equal to or even smaller than $2(\pi - a)$. For the sake of simplicity and without any loss of generality, the process is assumed to be sampled at a unit rate. The

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linear reconstruction of $Z(t)$ from the observed sample sequence $\{Z(n), n \in \mathbb{Z}\}$ can be obtained by

$$Z(t) = \sum_{n\in\mathbb{Z}} \frac{\sin \pi(t-n)}{\pi(t-n)} Z(n), \quad t \in \mathbb{R}, \tag{2}$$

with a zero mean square error. Eq. (2), referred to as the classical 'sampling formula', is attributed to several scientists (particularly to Shannon in the 1950s and to Lloyd for stationary processes [3]). The reconstruction formula (2) yields a linear reconstruction in the form of an infinite sum of interpolation functions weighted by sample values. In the oversampled case, other formulas with better convergence can be established using different interpolation functions such as raised cosine functions [4]. Besides classical uniform sampling, many sampling schemes have been considered in the literature (see for instance [5–7] and references therein). Particularly, multiperiodic sampling has been studied as partitions of the sampling sequence in a finite number of periodic subsequences [8–11].

This paper proposes new formulas with an arbitrarily tunable convergence rate in the case of oversampled band-limited stationary processes or functions. The proposed sampling scheme can be related to multiperiodic sampling: the reconstruction use a multiperiodic sample subsequence combined with particular isolated samples. The proposed reconstruction formulas are of the form of (2) but some samples are omitted. Consequently, the formulas can be easily fitted to any sample loss or deterioration by an appropriate time index translation. The reconstruction accuracy can be characterized at the omitted sample points. The choice of specific interpolation functions, involving trigonometric and polynomial functions, result in an exact reconstruction even in the neighbourhood of any lost sample. A general reconstruction formula expression is proposed. Specific expressions are derived for given process bandwidth ranges. A given process spectral occupancy leads to a set of reconstruction formulas characterized by the same periodic sample subsequences and increasing polynomial orders. The convergence rate improves when the polynomial order increases.

General formulas are proposed in Section 2. Section 3 gives particular expressions for given spectral occupancies of the observed process and reconstruction for small time values. The formulas can be easily adapted to different reconstruction time values. Section 4 studies the mean square convergence of the new reconstruction formulas and illustrates the new formula behaviour when a sample is lost or deteriorated. The mathematical proofs given in appendices are essentially based on complex integration theory.

2. New reconstruction formulas

Before going through the general expression of the proposed reconstruction formulas, some subsequences of the sample time set as well as some specific interpolation functions have to be defined.

2.1. Multiperiodic sample subsequence

For a unit sample rate, the sample time set is the set of relative integers $\mathbb{Z}$. The reconstruction formulas proposed in this paper use some subsequences of this sample time set. For a given positive integer Q, let define the following sample time subsequence:

$$J_{kl}^{Q} = Q^k\mathbb{Z} + (lQ^{k-1} - 1), \quad k \in \mathbb{N}^*, \ l = 1, 2, \ldots, Q-1. \tag{3}$$

In this paper, $Q = 2$ for simplicity but reconstruction formulas can be easily derived for other values of Q. Let J_k denote a subsequence obtained for $Q = 2$:

$$J_k = 2^k\mathbb{Z} + (2^{k-1} - 1), \quad k \in \mathbb{N}^*. \tag{4}$$

Thus, for instance:

- J_1 is the sequence of even integers $\{\ldots, -4, -2, 0, 2, 4, \ldots\}$,
- J_2 is the sequence of integers of the form $4m + 1$, $m \in \mathbb{Z}, \{\ldots, -7, -3, 1, 5, 9, \ldots\}, \ldots$

Note that these subsequences are disjoint but do not realize a partition of $\mathbb{Z}$ as shown in Appendix A:

$$J_k \cap J_l = \emptyset \quad \text{for } k \neq l \quad \text{and} \quad \bigcup_{k=1}^{\infty} J_k = \mathbb{Z} - \{-1\}. \tag{5}$$

In what follows, elements of J_k will be denoted by t_{mk}:

$$t_{mk} = 2^k m + (2^{k-1} - 1), \quad m \in \mathbb{Z}, \ k \in \mathbb{N}^*. \tag{6}$$

For a given process or function bandwidth, the reconstruction formulas use the N first subsequences $J_1, J_2, \ldots, J_N$ where N denotes a positive integer related to $\mathbf{Z}$ bandwidth through the parameter a by

$$N \geqslant N_{\min} = \inf\left\{ n \in \mathbb{N}^*; \sum_{k=1}^{n} 2^{-k} > 1 - \frac{a}{\pi} \right\}. \tag{7}$$