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A new approach for multistep numerical methods in several frequencies for perturbed oscillators

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ABSTRACT

This article presents a new multi-step numerical method based on φ-function series and designed to integrate forced oscillators with precision. The new algorithm retains the good properties of the MDF$_p$PC methods while presenting the advantages of greater precision and that of integrating the non-perturbed problem without any discretization error. In addition, this new method permits a single formulation to be obtained from the MDF$_p$PC schemes independently of the parity of the number of steps, which facilitates the design of a computational algorithm thus permitting improved implementation in a computer.

The construction of a new method for accurately integrating the homogenous problem is necessary if a method is sought which would be comparable to the methods based on Scheifele *G*-function series, very often used when problems of satellite orbital dynamics need to be resolved without discretization error.

Greater precision compared to the MDF$_p$PC methods and other known integrators is demonstrated by overcoming stiff and highly oscillatory problems with the new method and comparing approximations obtained with those calculated by means of other integrators.

1. Introduction

Forced oscillators are presented in many Physics and Engineering models. Harmonic oscillators also form part of Celestial Mechanics models, such as the classic two-body problem, and the satellite problem, as the K-S and B-F transformations reduce the Kepler problem to oscillators. Efficient numerical algorithms are needed, therefore, to provide very precise approximations.

In [1], the so called Ferrándiz φ- functions are defined, according to their properties. Based on these, a series method is constructed in order to numerically integrate perturbed, stable and convergent oscillators. Said method is a generalisation of the Scheifele series methods [2,3] which, with two frequencies integrates the non-perturbed problem exactly. Recently a new method, TFSTS, based on the Scheifele methods, which verify this property has been published in [4]. The series method is extremely precise. However, it has the disadvantage that it needs to be adapted to each specific problem. In order to resolve this difficulty, in [5], the transformation of the φ-function series method in the MDF$_p$PC multi-step scheme is explained. Calculation of the coefficients of the multi-step scheme are made through a recurrent procedure based on the existing relation between the divided differences and the elemental and complete symmetrical functions.

The recurrent calculation of the coefficients permits the MDF$_p$PC scheme to be considered as a VSVO type scheme.

The algorithm proposed in [5], despite its good behaviour, presents the difficulty of imprecise integration of the homogenous problem. In addition, it requires the definition of two multi-step schemes both in the explicit and implicit case, according to the parity of the number of steps, a fact which makes it difficult to implement in a computer.

This article introduces a new proposal for the MDF$_p$E, MDF$_p$I and MDF$_p$PC schemes in order to obtain a single formulation of the algorithms independently of the parity of the number of steps.

The new algorithm, while retaining the excellent properties of the aforementioned dual frequency methods, improves its precision, which is demonstrated by overcoming the problems proposed in [5] using both multi-step methods.

2. φ-functions series method

In this point a brief description is provided of the construction of the φ-functions and the corresponding numerical series method for integration of perturbed oscillators [1,5].

With $x(t)$ being the solution of the perturbed oscillator of the equation

$$x''(t) + \alpha^2 x(t) = \varepsilon \cdot f(x(t), x'(t), t),$$
$$x(0) = x_0, x'(0) = x'_0, \tag{1}$$

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corresponding to a forced oscillation with frequency α^2 and perturbation function $f = f(x(t), x'(t), t)$, from which it is supposed that it is analytic and with continuous partial derivatives with respect to the independent variables x, x', t in $I = [-T, T]$. Similarly it is admitted that the solution $x(t; x_0, x'_0, t_0)$ obtained with the initial conditions given is also analytic in I.

Under these conditions, the perturbation function may be expressed throughout the solution as $g(t) = f(x(t; x_0, x'_0, t_0), x'(t; x_0, x'_0, t_0), t)$.

Applying the differential operator $D^2 + \beta^2$ to IVP (1),

$$D^4x + (\alpha^2 + \beta^2)D^2x + \alpha^2\beta^2 x = (D^2 + \beta^2)\varepsilon g(t), \text{ is obtained.} \tag{2}$$

Taking into account the initial conditions and the form of the IVP (1)

$$\begin{aligned} &x(0) = x_0, \quad x'(0) = x'_0, \text{ is obtained} \\ &x''(0) = -\alpha^2 x_0 + \varepsilon f(x_0, x'_0, 0) = x''_0, \\ &x'''(0) = -\alpha^2 x'_0 + \varepsilon \vec{\nabla} f(x_0, x'_0, 0) \cdot (x'_0, x''_0, 1) = x'''_0 \end{aligned} \tag{3}$$

where $\vec{\nabla} f$ is the usual notation of the gradient vector

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial x'}, \frac{\partial f}{\partial t}\right).$$

Introducing the notation $L_4 \ (x) = (D^2 + \beta^2)(D^2 + \alpha^2)x$, and expressing $g(t) = \sum_{n=0}^{\infty} c_n \frac{t^n}{n!}$, a new IVP is obtained

$$\begin{aligned} &L_4(x) = \varepsilon \sum_{n=0}^{\infty} (c_{n+2} + \beta^2 c_n)\frac{t^n}{n!} \\ &x(0) = x_0, \quad x'(0) = x'_0, \quad x''(0) = x''_0, \quad x'''(0) = x'''_0, \end{aligned} \tag{4}$$

with the same exact solution $x(t; x_0, x'_0, t_0)$ as the original problem (1) in $I = [-T, T]$. This solution will be expressed as the sum of the solution to the non-perturbed problem with the initial conditions given, plus the solution of the perturbed problem with the initial null conditions. In order to obtain the solution to the perturbed problem, the principle of superposition of solutions is applied to the following family of IVPs:

$$\begin{aligned} &L_4(x_n) = \frac{t^n}{n!} \\ &x_n(0) = x'_n(0) = x''_n(0) = x'''_n(0) = 0. \end{aligned} \tag{5}$$

Solutions of (5) are denoted by $\Psi_n(t)$ with

$$n \geqslant 0. \tag{6}$$

The functions $\Psi_n(t)$, depend on α and β satisfying:

$$\Psi'_n(t) = \Psi_{n-1}(t) \text{ with } n \geqslant 1 \text{ and} \tag{7}$$

$$\Psi_n(t) + (\alpha^2 + \beta^2)\Psi_{n+2}(t) + \alpha^2\beta^2\Psi_{n+4}(t) = \frac{t^{n+4}}{(n+4)!} \tag{8}$$

with $n \geqslant 0$.

Depending on the values α and β it is possible to consider the five following cases:

2.1. Case I. ($\alpha \neq 0, \beta \neq 0, \alpha \neq \beta$)

In order to obtain the solution to the non-perturbed problem with the initial conditions given, the following "canonical" IVP's are resolved:

$$L_4(\varphi_i(t)) = 0 \tag{9}$$

$\varphi_i^{(j)}(0) = \delta_{i,j} \quad i,j = 0,1,2,3.$ with $\delta_{i,j}$ being the Kronecker delta.

The solution to the non-perturbed problem with the initial conditions given is

$$x_H(t) = x_0\varphi_0(t) + x'_0\varphi_1(t) + x''_0\varphi_2(t) + x'''_0\varphi_3(t) \tag{10}$$

where

$$\begin{aligned} &\varphi_0(t) = \frac{1}{\alpha^2 - \beta^2}(\alpha^2\cos(\beta t) - \beta^2\cos(\alpha t)) \\ &\varphi_1(t) = \frac{1}{\alpha^2 - \beta^2}\left(\frac{\alpha^2}{\beta}\sin(\beta t) - \frac{\beta^2}{\alpha}\sin(\alpha t)\right) \\ &\varphi_2(t) = \frac{1}{\alpha^2 - \beta^2}(\cos(\beta t) - \cos(\alpha t)) \\ &\varphi_3(t) = \frac{1}{\alpha^2 - \beta^2}\left(\frac{1}{\beta^2}\sin(\beta t) - \frac{1}{\alpha^2}\sin(\alpha t)\right). \end{aligned} \tag{11}$$

Therefore the solution of the IVP (4) is

$$x(t) = x_0\varphi_0(t) + x'_0\varphi_1(t) + x''_0\varphi_2(t) + x'''_0\varphi_3(t) + \varepsilon\sum_{n=0}^{\infty}(c_{n+2} + \beta^2 c_n)\Psi_n(t), \tag{12}$$

where

$$\Psi_n(t) = \sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+4+n)!}\frac{\beta^{2m+2} - \alpha^{2m+2}}{\beta^2 - \alpha^2}t^{2m+4+n}. \tag{13}$$

Defining

$$\varphi_{n+4}(t) = \Psi_n(t) \text{ with } n \geqslant 0, \tag{14}$$

it is possible to write the general solution of (4), in terms of the so-called Ferrandiz φ-functions.

$$\begin{aligned} x(t) = {} & x_0\varphi_0(t) + x'_0\varphi_1(t) + x''_0\varphi_2(t) + x'''_0\varphi_3(t) \\ & + \varepsilon\sum_{n=0}^{\infty}(c_{n+2} + \beta^2 c_n)\varphi_{n+4}(t), \end{aligned} \tag{15}$$

$$\begin{aligned} x'(t) = {} & x_0\varphi'_0(t) + x'_0\varphi_0(t) + x''_0\varphi'_2(t) + x'''_0\varphi_2(t) \\ & + \varepsilon\sum_{n=0}^{\infty}(c_{n+2} + \beta^2 c_n)\varphi_{n+3}(t). \end{aligned} \tag{16}$$

Let us suppose that we have calculated an approximation to the solution and its derivates in the point $t = nh$, we shall call these approximations x_n, x'_n, x''_n and x'''_n.

In order to obtain an approximation to the solution of

$$\begin{aligned} &L_4(x(t)) = \varepsilon f(x(t), x'(t), t) \\ &x(nh) = x_n, \quad x'(nh) = x'_n, \quad x''(nh) = x''_n, \quad x'''(nh) = x'''_n. \end{aligned} \tag{17}$$

in the point $(n+1)h$, in (17), the change was made to the independent variable $t = \tau + nh$, becoming

$$\begin{aligned} &L_4(x(\tau)) = \varepsilon f(x(\tau), x'(\tau), \tau) \\ &x(0) = x_n, \quad x'(0) = x'_n, \quad x''(0) = x''_n, \quad x'''(0) = x'''_n, \end{aligned} \tag{18}$$

thus making it possible to reinitialise the process, taking into account that

$$f(x(\tau), x'(\tau), \tau) = g(\tau) = \sum_{n=0}^{\infty} c_n\frac{\tau^n}{n!},$$

where

$$c_k = \frac{d^k g(0)}{d\tau^k} = \frac{d^k g(nh)}{dt^k}. \tag{19}$$

The approximation to the solution and its derivate in point $(n+1)h$, is given by

$$\begin{aligned} x_{n+1} = {} & x_n\varphi_0(h) + x'_n\varphi_1(h) + x''_n\varphi_2(h) + x'''_n\varphi_3(h) \\ & + \varepsilon\sum_{j=0}^{p-3}(c_{j+2} + \beta^2 c_j)\varphi_{j+4}(h) + \varepsilon\beta^2(c_{p-2}\varphi_{p+2} + c_{p-1}\varphi_{p+3}). \end{aligned} \tag{20}$$

$$\begin{aligned} x'_{n+1} = {} & x_n\varphi'_0(h) + x'_n\varphi_0(h) + x''_n\varphi'_2(h) + x'''_n\varphi_2(h) \\ & + \varepsilon\sum_{j=0}^{p-3}(c_{j+2} + \beta^2 c_j)\varphi_{j+3}(h) + \varepsilon\beta^2(c_{p-2}\varphi_{p+1} + c_{p-1}\varphi_{p+2}). \end{aligned} \tag{21}$$