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Quaternion Wigner–Ville distribution associated with the linear canonical transforms

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ABSTRACT

The quaternion linear canonical transform (QLCT), a generalization of the classical 2D Fourier transform, has gained much popularity in recent years because of its applications in many areas, including color image and signal processing. There are relationship between Wigner distribution and ambiguity function. But, these relations are only suitable for complex-valued signals, and have not been investigated in quaternion linear canonical transforms. The purpose of this paper is to propose an equivalent relationship for the quaternion Wigner distribution and quaternion ambiguity function in the QLCT setting. First, we propose the 2D quaternion Wigner distribution (QLWD) and quaternion ambiguity function associated with the QLCT. Next, the relationship between these two novel concepts are derived. Moreover, the connection with the corresponding analytic signal are investigated. Examples with bandpass analytic signal illustrate the features of the proposed distributions. Finally a novel algorithm for the detection of quaternion-valued linear frequency-modulated signal is presented by using the proposed QLWD.

1. Intoduction

The linear canonical transform (LCT) is a family of linear integral transforms, which has found broad applications in signal processing and optics [1,8,9,17]. Many operations such as the Fourier transform (FT), fractional Fourier transform, Lorentz transform and scaling operations are the special cases of the LCT. The LCT can be seen as an effective processing tool for chirp signal analysis such as the parameter estimation, sampling progress for non-bandlimited signals with non-linear Fourier atoms [28] and the LCT filtering [12,34,38]. As shown in [2,37] the LCT has been successfully used to study the generalized Wigner distribution (WD) and ambiguity function (AF). They have been shown to be very useful in non-stationary signal processing [2,37,17]. The ambiguity function (AF) also plays an important role in the non-stationary signal analysis and processing theory in classical time-frequency distribution and has applied in many fields such as sonar technology, radar signal processing and optical information processing (refer to [19]). We can use the AF to calculate the distance between the target and the radar, the speed of the target, the distance resolution and radia velocity resolution. In [5,33], the AF associated with the LCT are discussed. In [36], the authors discussed the properties of generalized Wigner distribution and ambiguity function by using new integral transform.

A first definition of a 2D quaternion linear canonical transform (QLCT) was introduced in [21,22], which is discussed later in this paper, and the first application of a 2D quaternion linear canonical transform to multidimensional signal analysis was reported in [21,22] involving prolate spheroidal wave signals and uncertainty principles [35]. Due to the non-commutative property of multiplication of quaternions, there are mainly three various types of 2D quaternion linear canonical transform (QLCTs): Two-sided QLCTs, Left-sided QLCTs and Right-sided QLCTs (refer to [22]). The main goals of the present paper are to study the properties of the Two-sided QLCTs (TQLCTs) of 2D quaternionic signals and to derive the novel concept of quaternion Wigner distribution (QLWD) and quaternion ambiguity function (QLAF). The classical Wigner distribution is a kind of important tool in the time-frequency signal analysis (refer to [7,12]). It has advantages for the detection of linear frequency-modulated (LFM) signals, which realizes energy accumulation in the time-frequency domain. In [3], the authors introduced the 2D quaternion Wigner distribution by substituting the kernel of FT with the kernel of QFT in the classical WD definition.

The theory of functions with values in the Hamiltonian quaternion algebra has been developed for decades as a generalization of the holomorphic function theory in the complex plane to 3D

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and 4D, and considered as a refinement of classical harmonic analysis; see e.g. [13,14,26,27]. So far, quaternionic analysis has been successfully used in a wide variety of fields from theoretical to practical problems such as gauge theories, mathematical physics, signal and image processing, navigation, computer vision, robotics as well as natural sciences and engineering [14,13,27,39].

In quaternionic analysis, the quaternion Fourier transform (QFT) [6,16,29–31] plays a vital role in the representation of (multidimensional) signals. It transforms a real (or quaternionic) 2D signal into a quaternion-valued frequency domain signal. The four components of the QFT separate four cases of symmetry into real signals instead of only two as in the complex FT. In [32] the authors applied the QFT to proceed color image analysis. The efficient implementation of the QFT was studied in [31]. In [4] the authors applied the QFT to image preprocessing and neural computing techniques for speech recognition. The QLCT, as the generalization of the QFT, was firstly studied in [22]. In this paper [22] the authors investigated the uncertainty principle of the (right-sided) QLCT within quaternionic analysis. From a signal processing point of view, this principle prescribes a lower bound on the product of the effective widths of quaternionic signals in the spatial and frequency domains. It is shown that only a 2D Gaussian signal minimizes the uncertainty. Recently, certain asymptotic properties of the QFT were analyzed and quaternionic counterparts of classical Bochner–Minlos theorems were derived in [10] and [11]. In [23] authors established a generalized Riemann–Lebesgue lemma for the (right-sided) QLCT, which prescribes the asymptotic behavior of the QLCT extending and refining the classical Riemann–Lebesgue lemma for the Fourier transform of 2D quaternion signals. The QLCT of a probability measure was introduced and some of its basic properties such as linearity, reconstruction formula, continuity, boundedness, and positivity were studied [25]. Different approaches of the 2D quaternion Hilbert transforms [24] are proposed recently which allow the calculation of the associated analytic signals in the QLCT domains. The corresponding analytic signals can suppress the negative frequency components in the QLCT domains. Envelope detection for color images are investigated [24] by the generalized analytic signals in the QLCT domains [24].

The paper is organized as follows. In order to make it self-contained, Section 2 gives a brief introduction to some general definitions and basic properties of quaternion algebra, LCTs and QLCTs of 2D Quaternion-valued signals. In Section 3 we give the definition and study the properties of a 2D QLWD and QLAF associated with QLCT. The relationship between Two-sided QLWD and QLAF is presented in Section 4. The proposed Two-sided QLWD combining with analytic signal is studied in Section 5. Examples with bandpass analytic signal illustrate the features of the proposed distributions. A novel algorithm for the detection of quaternion-valued linear frequency-modulated signal is presented in Section 6. Finally, concluding remarks are drawn in Section 7.

2. Preliminaries

2.1. The quaternion algebra

The quaternion algebra was introduced by Hamilton in 1843 and is denoted by $\mathbb{H}$ in his honor. Every element of $\mathbb{H}$ is a linear combination of a real scalar and three orthogonal imaginary units (denoted by $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$) with real coefficients (see [22])

$$\mathbb{H} = \{q \mid q := [q]_0 + \mathbf{i}[q]_1 + \mathbf{j}[q]_2 + \mathbf{k}[q]_3,\ [q]_i \in \mathbb{R},\ i = 0, 1, 2, 3\}, \tag{1}$$

where the elements $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ obey Hamilton's multiplication rules

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}. \tag{2}$$

Let $[q]_0$ and $\underline{q} := \mathbf{i}[q]_1 + \mathbf{j}[q]_2 + \mathbf{k}[q]_3$ denote the real scalar part and the vector part of quaternion number $q = [q]_0 + \mathbf{i}[q]_1 + \mathbf{j}[q]_2 + \mathbf{k}[q]_3$, respectively. Then, from [16], the real scalar part has a cyclic multiplication symmetry

$$[pqr]_0 = [qrp]_0 = [rpq]_0, \quad \forall\, p, q, r \in \mathbb{H}. \tag{3}$$

The conjugate of a quaternion q is defined by $\bar{q} = [q]_0 - \mathbf{i}[q]_1 - \mathbf{j}[q]_2 - \mathbf{k}[q]_3$, and the norm of $q \in \mathbb{H}$ defined as

$$|q| := \sqrt{q\bar{q}} = \sqrt{[q]_0^2 + [q]_1^2 + [q]_2^2 + [q]_3^2}. \tag{4}$$

It is easy to verify that

$$\overline{pq} = \bar{q}\bar{p}, \quad |qp| = |q||p|, \quad \forall\, p, q \in \mathbb{H}. \tag{5}$$

Now we introduce an inner product of quaternion functions f, g defined on $\mathbb{R}^2$ with values in $\mathbb{H}$ as follows

$$(f, g) := \int_{\mathbb{R}^2} f(\mathbf{x})\overline{g(\mathbf{x})}\, d^2\mathbf{x}, \quad d^2\mathbf{x} = dxdy, \tag{6}$$

with symmetric real scalar part

$$\langle f, g\rangle := \frac{1}{2}\left\{(f, g) + (g, f)\right\} = \int_{\mathbb{R}^2} [f(\mathbf{x})\overline{g(\mathbf{x})}]_0\, d^2\mathbf{x}. \tag{7}$$

Its associated scalar norm $\|f\|$ can be defined by both (6) and (7):

$$\|f\|_{L^2(\mathbb{R}^2;\mathbb{H})} := \sqrt{(f, f)} = \sqrt{\langle f, f\rangle} = \left(\int_{\mathbb{R}^2} |f(\mathbf{x})|^2 d^2\mathbf{x}\right)^{1/2}. \tag{8}$$

As a consequence of the inner product (6) we obtain the quaternion Cauchy–Schwarz inequality

$$\left|\int_{\mathbb{R}^2} f(\mathbf{x})\overline{g(\mathbf{x})}\, d^2\mathbf{x}\right| \le \left(\int_{\mathbb{R}^2} |f(\mathbf{x})|^2 d^2\mathbf{x}\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} |g(\mathbf{x})|^2 d^2\mathbf{x}\right)^{\frac{1}{2}}, \tag{9}$$

for any $f, g \in L^2(\mathbb{R}^2; \mathbb{H})$.

2.2. LCTs and QLCTs of 2D quaternion-valued signals

Due to the noncommutative property of multiplication of quaternions, there are various types of LCTs and QLCTs for 2D quaternion-valued signals. In the following we briefly recall the definitions of the LCT and QLCT (refer to [22,20]).

Definition 2.1 (*LCTs of 2D $\mathbb{H}$-valued signals*). Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2\times 2}$ be a matrix parameter such that $\det(A) = ad - bc = 1$. The Left-sided LCT and Right-sided LCT of signals $f \in L^1(\mathbb{R}^2; \mathbb{H})$ are defined by

$$\mathcal{L}^{\mathbf{i}}_{l,A}(f)(u_1, x_2) := \begin{cases} \frac{1}{\sqrt{\mathbf{i}2\pi b}} \int_{\mathbb{R}} e^{\mathbf{i}\left(\frac{a}{2b}x_1^2 - \frac{1}{b}x_1u_1 + \frac{d}{2b}u_1^2\right)} f(x_1, x_2)\, dx_1, & b \neq 0; \\ \sqrt{d}\, e^{\mathbf{i}\frac{cd}{2}u_1^2} f(du_1, x_2), & b = 0 \end{cases}$$

and

$$\mathcal{L}^{\mathbf{j}}_{r,A}(f)(x_1, u_2) := \begin{cases} \int_{\mathbb{R}} f(x_1, x_2) \frac{1}{\sqrt{\mathbf{j}2\pi b}} e^{\mathbf{j}\left(\frac{a}{2b}x_2^2 - \frac{1}{b}x_2u_2 + \frac{d}{2b}u_2^2\right)} dx_2, & b \neq 0; \\ f(x_1, du_2)\sqrt{d}\, e^{\mathbf{j}\frac{cd}{2}u_2^2}, & b = 0, \end{cases}$$

respectively.

Moreover, there are different types of QLCTs for 2D $\mathbb{H}$-valued signals.

Definition 2.2 (*QLCTs of 2D $\mathbb{H}$-valued signals*). Let $A_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in \mathbb{R}^{2\times 2}$ be a matrix parameter such that $\det(A_i) = 1$, $b_i \neq 0$