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Designs of matrix fractional order differentiators

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ABSTRACT

In this paper, the designs of matrix fractional order differentiator (MFOD) for differentiating digital signals are presented. First, the definitions of fractional derivatives are reviewed briefly and design problem of MFOD is stated. Then, three kinds of methods for designing MFOD are described including the conventional FIR and IIR filter methods, the discrete sine transform (DST) and discrete cosine transform (DCT) methods, and optimization methods. Next, numerical examples are demonstrated to compare the performances of these three design methods and the variable MFOD design is also studied. Finally, the image sharpening application and signal de-nosing application are used to show the effectiveness of the proposed matrix fractional order differentiators.

1. Introduction

In recent years, fractional order signal processing has received great attentions in many engineering applications. The research topics include fractional Fourier transform, fractional delay filter, fractional Hilbert transformer, fractional order moment and fractional calculus [1–5]. In the research area of fractional calculus, the integer order n of derivative $D^n x(t) = d^n x(t)/dt^n$ of function $x(t)$ is generalized to fractional order $D^v x(t)$, where v is a real number. So far, fractional differentiation and fractional derivative have extensively used in digital signal processing applications described below: First is the design of one-dimensional (1-D) and two-dimensional (2-D) digital FIR filters with fractional derivative constraints [6,7]. Second is the digital image processing in which fractional derivative is used to detect the edges [8], enhance the contrast of images [9] and reconstruct a higher resolution image from the associated lower resolution image [10]. Third is the signature verification in which fractional differential operator is applied to extract the dynamic feature from the handwritten signature [11]. Due to the success of the fractional calculus in signal processing, it is interesting to design various fractional order differentiators to solve the digital signal processing problems.

In order to compute the fractional derivatives of digital signals, several design methods of fractional order differentiators (FOD) have been presented. The ideal frequency response of conventional FOD is given by

$$H_d(\omega) = (j\omega)^v = \omega^v e^{j(v\pi/2)} \quad (1)$$

where $j = \sqrt{-1}$ and v is a real number. Thus, the FOD design problem is how to find a digital filter such that its actual frequency response fits the ideal response $H_d(\omega)$ as well as possible. When the order v is fixed, it is called the fixed fractional order differentiator (FFOD) design. So far, some methods have been proposed to design FFOD including the Taylor series expansion method [12], continued fraction expansion method [13], fractional sample delay method [14], radial basis function method [15] and discrete cosine transform method [16]. If the order v is adjustable, it is called the variable fractional order differentiator (VFOD) design. Several typical methods to solve the VFOD design problem are the weighted least squares method [11,17] and series expansion method [18,19]. All methods have their unique features.

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On the other hand, the matrix filters are particularly useful for filtering short data records. In [20], the convex optimization method is used to design low-pass filter, band-pass filter and Hilbert transformer. In [21], the matrix band-pass filter design using semi-infinite programming with application to direction-of-arrival (DOA) estimation is presented. In [22], a simple computer-aided approach for designing matrix low-pass filter is presented based on least-square criterion. Due to the success of matrix filter for filtering short data records, it is interesting to design matrix fractional order differentiators (MFOD) for computing the fractional derivatives of digital signals. Although the conventional FIR and IIR fractional order differentiators in [12–16] can be used to design MFOD, their performances are not good because of the causal and time-invariant constraints imposed on FIR and IIR FOD. To improve the performance, the discrete sine and cosine transform methods and optimization methods are presented to design MFOD in this paper. Because the closed-form and optimal solutions are obtained, the MFOD are easily used in various signal processing applications.

This paper is organized as follows. In Section 2, the definitions of fractional derivatives are reviewed briefly and design problem of MFOD is stated. In Section 3, the designs of MFOD using digital FIR and IIR FOD are presented. In Section 4, the designs of MFOD using discrete cosine and sine transforms are described. In Section 5, the designs of MFOD using optimizations method are studied including least squares (LS) method and convex optimization method. In Section 6, the variable MFOD is designed by using discrete sine transform and Taylor series expansion methods. The main feature of variable MFOD is that the fractional order ν can be quickly changed without re-designing a new MFOD. In Section 7, the digital image sharpening application and signal de-nosing application are used to show the effectiveness of the proposed matrix fractional order differentiators. Finally, a conclusion is made.

2. Fractional derivative and problem statement

In this section, the definition of fractional derivative is first reviewed briefly. Then, the design problem of matrix fractional order differentiator is stated.

2.1. Fractional derivative

In the literature, there are several definitions of fractional derivative and integral such as the Riemann–Liouville, the Grünwald–Letnikov and the Caputo definitions [5–7]. In this paper, we will use the Grünwald–Letnikov derivative whose definition is given by

$$D^{\nu}x(t)=\frac{d^{\nu}x(t)}{dt^{\nu}}=\lim_{\Delta\to 0}\sum_{k=0}^{\infty}\frac{(-1)^k C_k^{\nu}}{\Delta^{\nu}}x(t-k\Delta) \tag{2}$$

where coefficient C_k^{ν} is given by

$$C_k^{\nu}=\frac{\Gamma(\nu+1)}{\Gamma(k+1)\Gamma(\nu-k+1)}=\begin{cases}1, & k=0,\\ \frac{\nu(\nu-1)(\nu-2)\cdots(\nu-k+1)}{1.2.3\cdots k}, & k\geq 1\end{cases} \tag{3}$$

The above notation $\Gamma(\cdot)$ is the gamma function. Based on this definition, it can be shown that the fractional derivatives of exponential, power and trigonometric functions are given by

$$D^{\nu}e^{\alpha t}=\alpha^{\nu}e^{\alpha t} \tag{4a}$$

$$D^{\nu}t^{q}=\frac{\Gamma(q+1)}{\Gamma(q-\nu+1)}t^{q-\nu} \tag{4b}$$

$$D^{\nu}A\ \sin(\omega t+\phi)=A\omega^{\nu}\ \sin(\omega t+\phi+\tfrac{\pi}{2}\nu) \tag{4c}$$

$$D^{\nu}A\ \cos(\omega t+\phi)=A\omega^{\nu}\ \cos(\omega t+\phi+\tfrac{\pi}{2}\nu) \tag{4d}$$

So far, the definition of fractional derivative has been described.

2.2. Problem statement

Given a real-valued input data sequence $x(0)$, $x(1),\ldots,$ $x(N-1)$ whose elements are placed in a vector $\boldsymbol{x}$ and the filtered output sequence $y(0)$, $y(1),\ldots,$ $y(N-1)$ whose elements are placed in a vector $\boldsymbol{y}$, the matrix filtering operation can be expressed as

$$\boldsymbol{y}=\boldsymbol{H}\boldsymbol{x} \tag{5}$$

where $\boldsymbol{H}$ is an $N\times N$ matrix filter. Three properties of matrix filter in (5) are described below:

(1) If $\boldsymbol{y}_1=\boldsymbol{H}\boldsymbol{x}_1$ and $\boldsymbol{y}_2=\boldsymbol{H}\boldsymbol{x}_2$ are valid, then the expression $\alpha_1\boldsymbol{y}_1+\alpha_2\boldsymbol{y}_2=\boldsymbol{H}(\alpha_1\boldsymbol{x}_1+\alpha_2\boldsymbol{x}_2)$ holds. So, matrix filter is a linear filter.
(2) If matrix $\boldsymbol{H}$ is a Toeplitz matrix, then matrix filter is a time-invariant filter.
(3) If matrix $\boldsymbol{H}$ is a lower triangular matrix, then matrix filter is a causal filter because $y(n)$ only depends on the $x(n)$, $x(n-1),\ldots,$ $x(0)$.

In this paper, the design problem of matrix fractional order differentiator (MFOD) is studied. That is, the problem is how to determine the matrix $\boldsymbol{H}$ such that filter output $\boldsymbol{y}$ is the fractional derivative of input vector $\boldsymbol{x}$. Define the vector of the cosine sequence below:

$$\boldsymbol{a}(\omega)=[\ \cos(0\omega)\quad \cos(1\omega)\quad \cos(2\omega)\quad \cdots\quad \cos((N-1)\omega)]^T \tag{6}$$

then the ideal fractional derivative vector of $\boldsymbol{a}(\omega)$ is given by

$$\begin{aligned}\boldsymbol{b}(\omega)&=D^{\nu}\boldsymbol{a}(\omega)\\&=\left[\omega^{\nu}\ \cos(0\omega+\tfrac{\pi}{2}\nu)\quad \omega^{\nu}\ \cos(1\omega+\tfrac{\pi}{2}\nu)\quad\cdots\quad \omega^{\nu}\ \cos((N-1)\omega+\tfrac{\pi}{2}\nu)\right]^T\end{aligned} \tag{7}$$

Clearly, the amplitude of cosine sequence is amplified by ω^{ν} and phase is shifted by $\nu\pi/2$ which are consistent with the specification of the ideal frequency response $H_d(\omega)$ in (1). Therefore, the design problem reduces to how to determine the matrix $\boldsymbol{H}$ such that filter output $\boldsymbol{H}\boldsymbol{a}(\omega)$ approximates the ideal vector $\boldsymbol{b}(\omega)$ as well as possible for all ω in the interested frequency band R. In this paper, three kinds of design methods will be presented to find the matrix $\boldsymbol{H}$ including conventional FIR and IIR fractional order differentiator (FOD) methods, discrete sine transform (DST) and discrete cosine transform (DCT) methods, and optimization methods. The details will be described in next three sections. To evaluate performances of these design methods, an average integral squared error is defined as

$$E_{av}=\frac{\int_{\omega\in R}\|\boldsymbol{H}\boldsymbol{a}(\omega)-\boldsymbol{b}(\omega)\|_2^2\,d\omega}{N} \tag{8}$$

Thus, the smaller the error E_{av} is, the better the design method is.