Signal Processing 95 (2014) 49–57

Contents lists available at ScienceDirect

Signal Processing

journal homepage: www.elsevier.com/locate/sigpro

Testing blind separability of complex Gaussian mixtures

David Ramírez [a,*], Peter J. Schreier [a], Javier Vía [b], Ignacio Santamaría [b]

[a] Signal and System Theory Group, Universität Paderborn, Paderborn, Germany
[b] Communications Engineering Dept., University of Cantabria, Santander, Spain

ARTICLE INFO

Article history:
Received 11 February 2013
Received in revised form
23 June 2013
Accepted 22 August 2013
Available online 30 August 2013

Keywords:
Complex independent component analysis (ICA)
Circularity coefficients
Generalized likelihood ratio test (GLRT)
Hypothesis test
Maximum likelihood (ML) estimation
Wilks' theorem

ABSTRACT

The separation of a complex mixture based solely on second-order statistics can be achieved using the Strong Uncorrelating Transform (SUT) if and only if all sources have distinct circularity coefficients. However, in most problems we do not know the circularity coefficients, and they must be estimated from observed data. In this work, we propose a detector, based on the generalized likelihood ratio test (GLRT), to test the separability of a complex Gaussian mixture using the SUT. For the separable case (distinct circularity coefficients), the maximum likelihood (ML) estimates are straightforward. On the other hand, for the non-separable case (at least one circularity coefficient has multiplicity greater than one), the ML estimates are much more difficult to obtain. To set the threshold, we exploit Wilks' theorem, which gives the asymptotic distribution of the GLRT under the null hypothesis. Finally, numerical simulations show the good performance of the proposed detector and the accuracy of Wilks' approximation.

1. Introduction

The blind separation of a linear mixture of complex independent sources is an important problem with a range of applications, e.g. in biomedical image analysis. See [1–3], and references therein. The Strong Uncorrelating Transform (SUT) allows blind separation based solely on second-order statistics [4–6], provided that these sources correlate with their complex conjugates and that the strengths of these correlations differ from source to source. A complex random variable x that correlates with its complex conjugate x^* has a nonzero complementary covariance $E[x^2]$ and is called improper or noncircular.

The circularity coefficient $k = |E[x^2]|/E[|x|^2]$ takes values between 0 and 1 and measures how noncircular or improper a random variable is. This may be illustrated by the density contours of a univariate complex Gaussian random variable. These contours are ellipses, and the shape of these ellipses is controlled by the circularity coefficients [7]. If a Gaussian random variable has circularity coefficient $k=0$, then its probability density contours are circular [8–10]; if it has circularity coefficient $k=1$, then its probability density contours degenerate into a line in the complex plane.

The circularity coefficients are invariant to linear transformations. Thus, a linear mixture of complex sources has the same set of circularity coefficients as the original sources. This invariance property is exploited by the SUT for blind separation. A necessary and sufficient condition for separability using the SUT is that all circularity coefficients of the sources are distinct. It thus makes intuitive sense that separation of the mixture should be easier if the circularity coefficients are more clearly separated, and it should become more difficult if the circularity coefficients are more clustered. This intuition is supported theoretically by [11].

In practice, the circularity coefficients are not known a priori and must be estimated from the observed data.

* Corresponding author. Tel.: +49 525 160 4367.
E-mail addresses: david.ramirez@sst.upb.de (D. Ramírez), peter.schreier@sst.upb.de (P.J. Schreier), jvia@gtas.dicom.unican.es (J. Vía), nacho@gtas.dicom.unican.es (I. Santamaría).

0165-1684/$ - see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.sigpro.2013.08.010

We are thus confronted with the question whether or not a mixture is separable, based on a given set of observations. This paper deals with this problem by deriving a generalized likelihood ratio test (GLRT) to decide whether a mixture of complex-valued improper signals is separable or not. The test boils down to testing whether all circularity coefficients are distinct or whether there are circularity coefficients with multiplicity greater than one. This paper extends preliminary results reported at a conference [12], where we did not include any details of the rather lengthy proofs.

The structure of our paper is as follows. In Section 2, we review how the SUT enables ICA of complex sources. In Section 3, we formally define our hypothesis testing problem, and in Section 4, we derive the GLRT. Finally, Section 5 presents simulation results that illustrate the performance of our detector.

1.1. Notation

In this paper we use bold-face upper-case letters to denote matrices, with elements $x_{k,l}$ or $[\mathbf{X}]_{k,l}$; bold-face lower-case letters for column vectors, and light-face lower case letters for scalar quantities. The superscripts $(\cdot)^T$ and $(\cdot)^H$ denote transpose and Hermitian transpose, respectively. The determinant and trace of a matrix $\mathbf{A}$ will be denoted, respectively, as $\det(\mathbf{A})$ and $\text{tr}(\mathbf{A})$. The notation $\mathbf{A} \in \mathbb{C}^{M\times N}$ ($\mathbf{A} \in \mathbb{R}^{M\times N}$) will be used to denote that $\mathbf{A}$ is a complex (real) matrix of dimension $M \times N$. For vectors, the notation $\mathbf{x} \in \mathbb{C}^M$ ($\mathbf{x} \in \mathbb{R}^M$) denotes that $\mathbf{x}$ is a complex (real) vector of dimension M, and $\mathbf{x} \sim \mathcal{CN}(\boldsymbol{\mu}, \mathbf{R})$ indicates that $\mathbf{x}$ is a complex circular Gaussian random vector of mean $\boldsymbol{\mu}$ and covariance matrix $\mathbf{R}$. The expectation operator will be denoted as $E[\cdot]$. The notation $\mathbf{I}_L$ is used to denote the identity matrix of size $L \times L$, whereas $\mathbf{I}_{L\times P}$ is a $L \times P$ matrix with ones in the main diagonal and zeros elsewhere. The matrix $\mathbf{0}_{L\times P}$ denotes the zero matrix of size $L \times P$. We use $\mathbf{A}^{1/2}$ to denote the positive semidefinite square root matrix of the positive semidefinite matrix $\mathbf{A}$. Finally, $\text{diag}(\mathbf{A})$ is a diagonal matrix formed by the main diagonal of $\mathbf{A}$ and $\text{diag}(\mathbf{a})$ is a diagonal matrix formed by the vector $\mathbf{a}$.

2. ICA from second-order statistics

In this section, we present a review of independent component analysis (ICA) of complex sources based solely on second-order statistics (SOS). This technique is based on the SUT [4–6]. Let us consider the instantaneous noiseless linear complex ICA model

$$\mathbf{x} = \mathbf{A}\mathbf{s}, \tag{1}$$

where $\mathbf{x} \in \mathbb{C}^P$ are the measurements, $\mathbf{A} \in \mathbb{C}^{P\times P}$ is the *unknown* mixing matrix, assumed to have full rank, and $\mathbf{s} \in \mathbb{C}^P$ are zero-mean sources, which are assumed to be independent. Note that there is the same number of sources and measurements. This is a safe assumption for overdetermined problems since we can always apply a dimensionality reduction technique based on principal component analysis (PCA). On the other hand, the case of fewer measurements than sources can be ignored since there exists no solution using only SOS.

The idea behind ICA is to recover $\mathbf{s}$ without knowledge of $\mathbf{A}$, utilizing only the linearity of the model and the independence of the sources. For the linear model (1), the sources are recovered as

$$\hat{\mathbf{s}} = \mathbf{B}\mathbf{x}, \tag{2}$$

where $\mathbf{B}$ is the separating matrix. Since the technique is based only on the independence of the sources, there exist some ambiguities. Any scaling of $\mathbf{s}$, i.e., multiplication with a diagonal matrix, and any reordering of the components of $\mathbf{s}$, i.e., multiplication with a permutation matrix, preserves independence. Hence, we can obtain $\mathbf{B}$ only up to a multiplication with a monomial matrix, which is the product of a permutation and a diagonal matrix.

Typically, ICA for real sources is based on higher-order statistics, and if there is more than one Gaussian source, it is only possible to recover $\mathbf{s}$ if the sources have some temporal (sample-to-sample) correlation with different autocorrelation functions [13]. Temporally uncorrelated complex sources, on the other hand, may be separated based on SOS, provided that these satisfy certain conditions. For complex random vectors, all the SOS information is contained in two matrices: the covariance matrix $\mathbf{R}_{ss} = E[\mathbf{s}\mathbf{s}^H]$ and the complementary covariance matrix $\tilde{\mathbf{R}}_{ss} = E[\mathbf{s}\mathbf{s}^T]$ [6]. The assumption of independent sources implies a diagonal structure for both the covariance matrix $\mathbf{R}_{ss}$ and the complementary covariance matrix $\tilde{\mathbf{R}}_{ss}$. Moreover, taking into account the ambiguities of the ICA problem, we may even make the stronger assumptions that $\mathbf{R}_{ss} = \mathbf{I}$ and $\tilde{\mathbf{R}}_{ss} = \mathbf{K}$, where $\mathbf{K} = \text{diag}(k_1, \ldots, k_P)$ and $1 \geq k_1 \geq \cdots \geq k_P \geq 0$. The diagonal elements k_i are the so-called *circularity coefficients* [5], which we will derive momentarily. Under these assumptions, the covariance matrix of the measurements is

$$\mathbf{R}_{xx} = E[\mathbf{x}\mathbf{x}^H] = \mathbf{A}\mathbf{R}_{ss}\mathbf{A}^H = \mathbf{A}\mathbf{A}^H, \tag{3}$$

and the complementary covariance matrix is

$$\tilde{\mathbf{R}}_{xx} = E[\mathbf{x}\mathbf{x}^T] = \mathbf{A}\tilde{\mathbf{R}}_{ss}\mathbf{A}^T = \mathbf{A}\mathbf{K}\mathbf{A}^T. \tag{4}$$

To recover $\mathbf{s}$, the separating matrix $\mathbf{B}$ must simultaneously diagonalize $\mathbf{R}_{xx}$ and $\tilde{\mathbf{R}}_{xx}$, i.e., both $\mathbf{B}\mathbf{R}_{xx}\mathbf{B}^H$ and $\mathbf{B}\tilde{\mathbf{R}}_{xx}\mathbf{B}^T$ must be diagonal. To this end, we first compute the coherence matrix

$$\mathbf{C} = \mathbf{R}_{xx}^{-1/2}\tilde{\mathbf{R}}_{xx}(\mathbf{R}_{xx}^*)^{-H/2} = \mathbf{R}_{xx}^{-1/2}\tilde{\mathbf{R}}_{xx}\mathbf{R}_{xx}^{-T/2}, \tag{5}$$

which appears in the canonical correlation analysis (CCA) [14] of the vectors $\mathbf{x}$ and $\mathbf{x}^*$ [6]. Then we obtain the Takagi factorization [15] of $\mathbf{C}$, which is a special singular value decomposition for complex and symmetric (not Hermitian symmetric) matrices $\mathbf{C} = \mathbf{C}^T$:

$$\mathbf{C} = \mathbf{F}\mathbf{K}\mathbf{F}^T, \tag{6}$$

where $\mathbf{F} \in \mathbb{C}^{P\times P}$ is a unitary matrix and $\mathbf{K} = \text{diag}(k_1, \ldots, k_P)$ is a diagonal matrix that contains the circularity coefficients. These circularity coefficients are the canonical correlations between $\mathbf{x}$ and $\mathbf{x}^*$. The separating matrix is now given by the SUT

$$\mathbf{B} = \mathbf{F}^H\mathbf{R}_{xx}^{-1/2}. \tag{7}$$

The complex ICA model is separable if and only if all circularity coefficients are distinct [4,5]. Hence, it is possible to