Signal Processing 95 (2014) 76–87

Contents lists available at ScienceDirect

Signal Processing

journal homepage: www.elsevier.com/locate/sigpro

A new class of almost symmetric orthogonal Hilbert pair of wavelets

Selvaraaju Murugesan, David B.H. Tay*

Department of Electronic Engineering, La Trobe University, Bundoora, Victoria 3086, Australia

ARTICLE INFO

Article history:
Received 31 May 2013
Received in revised form
22 August 2013
Accepted 26 August 2013
Available online 4 September 2013

Keywords:
Orthogonal filter banks
Hilbert pair
Wavelet transform

ABSTRACT

The dual-tree complex wavelet transform offers near shift invariance and a better directional selectivity compared to traditional discrete wavelet transforms. A new class of Hilbert-pair of wavelets that can be used in the dual-tree is presented in this work. These Hilbert-pairs are exactly orthogonal but are also almost symmetric. They therefore have the advantages found in both orthogonal and biorthogonal wavelets. Symmetry in the wavelets is of prime importance in many applications as it offers a better directional selectivity. An efficient and flexible design technique is proposed for the design of these new Hilbert-pairs. The proposed technique readily allows the designer to trade-off between the degree of symmetry and the analytic quality. The designed wavelet filters have good frequency response, flat group delay and achieve a good approximation to the half-sample delay condition which is required for good analytic quality.

1. Introduction

The discrete wavelet transform (DWT) is an indispensable tool in a wide range of engineering and scientific applications [1–3] and can be implemented efficiently with filter banks (FB) [4,5]. The DWT however suffers from the problems of shift variance and the lack of directionality. A new generation of wavelet based transforms have been proposed to overcome these problems [6,7] and they are usually overcomplete or redundant [8]. The dual tree complex wavelet transform (DTCWT) introduced by Kingsbury has emerged as one of the most popular redundant transforms in a wide variety of applications [9–11]. The DTCWT has near shift invariance, provides directional selectivity in multidimensions and has lower redundancy compared to the undecimated DWT [10].

The building block of DTCWT is a pair of two-channel perfect reconstruction (PR) multirate filter banks. The equivalent wavelet function of the two filter banks, $\psi^h(t)$ and $\psi^g(t)$ (with Fourier transform $\Psi^h(\omega)$ and $\Psi^g(\omega)$ respectively), should ideally satisfy the following Hilbert transform relationship:

$$\Psi^g(\omega) = \begin{cases} -j\Psi^h(\omega), & \omega > 0 \\ j\Psi^h(\omega), & \omega < 0 \end{cases} \quad (1)$$

The wavelets $(\psi^h(t), \psi^g(t))$ form a Hilbert-pair (and the same can be said of the corresponding pair of filter banks) and can be either biorthogonal or orthogonal. The near shift invariance and higher directional selectivity of DTCWT is due to (1) and the importance of this relationship is further explained in [12,13,10]. Reviews of earlier design techniques for Hilbert-pairs are found in [10,14]. Some of the more recent techniques are found in [15–20]. Most techniques proposed are for FIR filters but recently techniques for IIR filters have been proposed in [21,22].

Biorthogonal wavelets can be exactly symmetric and the design of biorthogonal Hilbert-pairs with symmetric linear phase odd/even length filters (Type I/II filters) can be found in [23–25]. Symmetry is particularly important in image processing as salient features such as lines and

* Corresponding author. Tel.: +61 394792529.
E-mail addresses: s.murugesan@latrobe.edu.au (S. Murugesan), d.tay@latrobe.edu.au (D.B.H. Tay).

0165-1684/$ - see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.sigpro.2013.08.016

edges are particularly susceptible to nonlinear distortions. In time series analysis, symmetry allows for the alignment of wavelet coefficients [26, Chapter 4]. Biorthogonal transforms however are not l^2 norm preserving. Orthogonal (strictly orthonormal) transforms are l^2 norm preserving and have the advantage of noise decorrelation (in denoising), simple bit allocation (in compression), and more generally, energy preservation in the transform coefficients. However dyadic orthogonal wavelets based on a real coefficient finite-impulse-response (FIR) filter cannot be exactly symmetric [27] (except for the simplest Haar wavelets). Orthogonal wavelets based on infinite-impulse-response (IIR) filters [28,21] or complex coefficient filters [22,29] can be symmetric but require more complex implementation. The two most common types of orthogonal Hilbert-pairs are (i) those based on the common factor technique [13,19,20] and (ii) those based on symmetric-self-Hilbertian (SSH) filters [30–32] (which includes the Q-shift filters [33,34,20]). All orthogonal real coefficient FIR Hilbert-pairs reported so far do not have the symmetry of the biorthogonal Hilbert-pairs. The symmetry of the wavelets $\psi^h(t)$ and $\psi^g(t)$ is important for directional selectivity [13].

In this paper we present the design of a new class of real coefficient FIR Hilbert-pairs that have the advantages of both biorthogonal and orthogonal wavelets. The wavelets are exactly orthogonal but are also almost symmetric. The corresponding filters are almost like the Type I/II filters but are exactly orthogonal and therefore l^2 norm preserving. The phase response is approximately linear and the impulse response is approximately symmetric. In our previous work [35] the design of single filter bank with Type II like property via direct optimization was presented. This paper not only extends the work to Type I like filters but also presents the design framework to match the two filters to give Hilbert-pairs with good analytic quality. The overview of the paper is as follows. Section 2 briefly reviews filter bank and wavelet fundamentals. The design of Hilbert pair based on almost symmetric filters is presented in Section 3. Techniques to improve the analyticity of the Hilbert pair are presented in Section 4. Discussion and comparison with the previous work are presented in Section 5. This paper concludes in Section 6.

2. Preliminaries

2.1. Filter bank and wavelet fundamentals

Let $H_0(z) \equiv \sum_{n=-(L-1)}^{L} h_0(n) z^{-n}$ be the (even) length $L_f = 2L$ analysis low pass filter of a two channel orthogonal filter bank (FB) and is also known as a CQF (conjugate quadrature filter). Note that an almost-centered-at-the-origin (ACO) version of the CQF with support $n \in [-(L-1), L]$ is considered for convenience. The synthesis low pass filter is given by $F_0(z) = z^{-1} H_0(z^{-1})$. The analysis and synthesis high pass filters H_1 and F_1 respectively are obtained as follows: $H_1(z) = z^{-1} H_0(-z^{-1})$, $F_1(z) = z H_0(-z)$. The perfect reconstruction (PR) condition is given by $P(z) + P(-z) = 2$ where $P(z) = H_0(z) H_0(z^{-1})$ is the product filter. The low-pass filter $H_0(z)$ can be obtained from the spectral factorization of a suitably designed product filter which must have non-negative frequency response $P(e^{j\omega}) \geq 0$. Ensuring non-negativity in the design process can be problematic and this is one of the major drawback with the product filter approach. An alternative approach is to use the lattice structure parametrization [5]:

$$\begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix} = z^{(L-1)} \boldsymbol{H}_p(z^2) \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix} \tag{2}$$

where $\boldsymbol{H}_p(z)$ is the (analysis) polyphase matrix given by

$$\boldsymbol{H}_p(z) = \frac{1}{\sqrt{K}} \hat{\boldsymbol{H}}_p(z) = \frac{1}{\sqrt{K}} \boldsymbol{R}_L \prod_{l=1}^{L-1} [\boldsymbol{D}(z) \boldsymbol{R}_l] \tag{3}$$

$$\boldsymbol{D}(z) = \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix}, \quad \boldsymbol{R}_l = \begin{bmatrix} 1 & -\alpha_l \\ \alpha_l & 1 \end{bmatrix}$$

and the normalization constant is

$$K \equiv \prod_{l=1}^{L} (1 + \alpha_l^2) > 0.$$

The un-normalized polyphase matrix is $\hat{\boldsymbol{H}}_p(z)$. With the lattice structure the perfect reconstruction (PR) and non-negativity conditions are structurally imposed, i.e. there are no restrictions on the value of the lattice parameters α_l's. For the convenience later on the un-normalized filter (corresponding to the un-normalized polyphase matrix) is defined as $\hat{H}_0(z) \equiv \sum_{n=-(L-1)}^{L} \hat{h}_0(n) z^{-n} = \sqrt{K} H_0(z)$.

The scaling and wavelet functions, $\phi(t)$ and $\psi(t)$ respectively, are obtained from the filter coefficients via the two-scale equations $\phi(t) = \sqrt{2} \sum_n h_0(n) \phi(2t-n)$ and $\psi(t) = \sqrt{2} \sum_n h_1(n) \phi(2t-n)$. To ensure smooth scaling and wavelet functions zeros at $z = -1$ are imposed on $H_0(z)$, and this is known as the (wavelet) vanishing moment (VM) condition [4]. Sum rules can be applied to the filter coefficients $h_0(n)$ or $\hat{h}_0(n)$ to ensure VMs. The general expression for the kth sum rule is given by

$$\hat{H}_0^{(k)}(e^{j\omega}) = \sum_n (-1)^n n^k \hat{h}_0(n) = 0 \tag{4}$$

For p VMs p sum rules are required, i.e. (4) for $k = 0, \ldots, p-1$. With the lattice parametrization the un-normalized filter coefficients $\hat{h}_0(n)$ ($\equiv \sqrt{K} h_0(n)$) can be expressed as multilinear functions of the lattice parameters (this can be verified by careful inspection of Eqs. (2) and (3)). Using the multilinear functions in the VMs condition (4) (which is linear in the coefficients $\hat{h}_0(n)$) results in multilinear constraint equations on the parameters

$$\sum_n C_{n,k} \left(\prod_{l=1}^{L} \alpha_l^{i_{l,n}} \right) = 0 \tag{5}$$

for $k = 0, \ldots, K-1$, where $i_{l,n} = 0$ or 1 and $C_{n,k}$ are integer constants. The details of derivation of (5) from (4) can be found in the proof of Lemma 1 in [36]. For example with the length-8 filter with k=3, the constraint equation is $\alpha_1 + 27\alpha_2 - 8\alpha_1\alpha_2 + 125\alpha_3 - 64\alpha_1\alpha_3 - 8\alpha_2\alpha_3 - 27\alpha_1\alpha_2\alpha_3 = 0$.

2.2. Dual-tree filter bank

The dual-tree complex wavelet transform is based on a pair of filter banks. The upper and lower tree filters are denoted by superscripts h and g respectively. It is proven in