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Some results on the Weiss–Weinstein bound for conditional and unconditional signal models in array processing

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ABSTRACT

In this paper, the Weiss–Weinstein bound is analyzed in the context of sources localization with a planar array of sensors. Both conditional and unconditional source signal models are studied. First, some results are given in the multiple sources context without specifying the structure of the steering matrix and of the noise covariance matrix. Moreover, the case of an uniform or Gaussian prior are analyzed. Second, these results are applied to the particular case of a single source for two kinds of array geometries: a non-uniform linear array (elevation only) and an arbitrary planar (azimuth and elevation) array.

1. Introduction

Sources localization problem has been widely investigated in the literature with many applications such as radar, sonar, medical imaging, etc. One of the objective is to estimate the direction-of-arrival (DOA) of the sources using an array of sensors.

In array processing, lower bounds on the mean square error are usually used as a benchmark to evaluate the ultimate performance of an estimator. There exist several lower bounds in the literature. Depending on the assumptions about the parameters of interest, there are three main kinds of lower bounds. When the parameters are assumed to be deterministic (unknown), the main lower bounds on the (local) mean square error used are the well known Cramér–Rao bound and the Barankin bound (more particularly their approximations [1–4]). When the parameters are assumed to be random with a known prior distribution, these lower bounds on the global mean square error are called Bayesian bounds [5]. Some typical families of Bayesian bounds are the Ziv–Zakai family [6–8] and the Weiss–Weinstein family [9–12]. Finally, when the parameter vector is made from both deterministic and random parameters, the so-called hybrid bounds have been developed [13–15].

Since the DOA estimation is a non-linear problem, the outliers effect can appear and the estimators mean square error exhibits three distinct behaviors depending on the number of snapshots and/or on the signal to noise ratio (SNR) [16]. At high SNR and/or for a high number of snapshots, *i.e.*, in the asymptotic region, the outliers effect can be neglected and the ultimate performance are described by the (classical/Bayesian/hybrid) Cramér–Rao bound. However, when the SNR and/or the number of snapshots decrease, the outliers effect lead to a quick increase of the mean square error: this is the so-called threshold effect. In this region, the behavior of the lower bounds are not the same. Some bounds, generally called global bounds (Barankin, Ziv–Zakai, Weiss–Weinstein) can predict the threshold while the others, called local bounds, like the Cramér–Rao bound or the Bhattacharyya bound cannot. Finally, at low SNR and/or at low number of snapshots, *i.e.*, in the

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no-information region, the deterministic bounds exceed the estimator mean square error due to the fact that they do not take into account the parameter support. On the contrary, the Bayesian bounds exploit the parameter prior information leading to a “real” lower bound on the global mean square error.

In this paper,[1] we are interested in the Weiss–Weinstein bounds which is known to be one of the tightest Bayesian bound with the bounds of the Ziv–Zakai family. We will study the two main source models used in the literature [17]: the unconditional (or stochastic) model where the source signals are assumed to be Gaussian and the conditional (or deterministic) model where the source signals are assumed to be deterministic. Surprisingly, in the context of array processing, while closed-form expressions of the Ziv–Zakai bound (more precisely its extension by Bell et. al. [18]) were proposed around 15 years ago for the unconditional model, the results concerning the Weiss–Weinstein bound are, most of the time, only conducted by way of computations. Concerning the unconditional model, in [19], the Weiss–Weinstein bound has been evaluated by way of computations and has been compared to the mean square error of the MUSIC algorithm and classical Beamforming using a particular 8×8 element array antenna. In [20], the authors have introduced a numerical comparison between the Bayesian Cramér–Rao bound, the Ziv–Zakai bound and the Weiss–Weinstein bound for DOA estimation. In [21], numerical computations of the Weiss–Weinstein bound to optimize sensor positions for non-uniform linear arrays have been presented. Again in the unconditional model context, in [22], by considering the matched-field estimation problem, the authors have derived a semi closed-form expression of a simplified version of the Weiss–Weinstein bound for the DOA estimation. Indeed, the integration over the prior probability density function was not performed. The conditional model (with known waveforms) is studied only in [23], where a closed-form expression of the WWB is given in the simple case of spectral analysis and in [24] which is a simplified version of the bound.

While the primary goal of this paper is to give closed-form expressions of the Weiss–Weinstein bound for the DOA estimation of a single source with an arbitrary planar array of sensors, under both conditional and unconditional source signal models, we also provide partial closed-form expressions of the bound which could be useful for other problems. First, we study the general Gaussian observation model with parameterized mean or parameterized covariance matrix. Indeed, one of the success of the Cramér–Rao is that, for this observation model, a closed-form expression of the Fisher information matrix is available: this is the so-called Slepian–Bang formula [25]. Such kind of formulas have been less investigated in the context of bounds tighter than the Cramér–Rao bound. Second, some results are given in the multiple sources context without specifying the structure of the steering matrix and of the noise covariance matrix. Finally, these results are applied to the particular case of a single source for two kinds of array geometries: the non-uniform linear array (elevation only) and the planar (azimuth and elevation) array. Consequently, the aim of this paper is also to provide a textbook of formulas which could be applied in other fields. The Weiss–Weinstein bound is known to depend on parameters called test points and other parameters generally denoted s_i. One particularity of this paper in comparison with the previous works on the Weiss–Weinstein bound is that we do not use the assumption $s_i = 1/2, \ \forall i$.

This paper is organized as follows. Section 2 is devoted to the array processing observation model which will be used in the paper. In Section 3, a short background on the Weiss–Weinstein bound is presented and two general closed-form expressions which will be the cornerstone for our array processing problems are derived. In Section 4 we apply these general results to the array processing problem without specifying the structure of the steering matrix. In Section 5, we study the particular case of the non-uniform linear array and of the planar array for which we provide both closed-form expressions of the bound in the context of a single stationary source in the far field area. Some simulation results are proposed in Section 6. Finally, Section 7 gives our conclusions.

2. Problem setup

In this section, the general observation model generally used in array signal processing is presented as well as the first different assumptions used in the remain of the paper. Particularly, the so-called conditional and unconditional source models are emphasized.

2.1. Observations model

We consider the classical scenario of an array with M sensors which receives N complex bandpass signals $\mathbf{s}(t) = [s_1(t)\ s_2(t) \dots s_N(t)]^T$. The output of the array is a $M \times 1$ complex vector $\mathbf{y}(t)$ which can be modelled as follows (see, *e.g.*, [26] or [17]):

$$\mathbf{y}(t) = \mathbf{A}(\boldsymbol{\theta})\mathbf{s}(t) + \mathbf{n}(t), \quad t = 1, \dots, T, \tag{1}$$

where T is the number of snapshots, where $\boldsymbol{\theta} = [\theta_1\ \theta_2 \dots \theta_q]^T$ is an unknown parameter vector of interest,[2] where $\mathbf{A}(\boldsymbol{\theta})$ is the so-called $M \times N$ steering matrix of the array response to the sources, and where the $M \times 1$ random vector $\mathbf{n}(t)$ is an additive noise.

[1] Section 5.2.2 of this paper has been partially presented in [24].

[2] Note that one source can be described by several parameters. Consequently, $q > N$ in general.