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Amplitudes of mono-component signals and the generalized sampling functions

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ABSTRACT

There is a recent trend to use *mono-components* to represent nonlinear and non-stationary signals rather than the usual Fourier basis with linear phase, such as the intrinsic mode functions used in Norden Huang's empirical mode decomposition [12]. A mono-component is a real-valued signal of finite energy that has non-negative instantaneous frequencies, which may be defined as the derivative of the phase function of the given real-valued signal through the approach of canonical amplitude-phase modulation. We study in this paper how the amplitude is determined by its phase for a class of signals, of which the instantaneous frequency is periodic and described by the Poisson kernel. Our finding is that such an amplitude can be perfectly represented by a sampling formula using the so-called generalized sampling functions that are related to the phase. The regularity of such an amplitude is identified to be at least continuous. Such characterization of mono-components provides the theory to adaptively decompose non-stationary signals. Meanwhile, we also make a very interesting and new characterization of the band-limited functions.

1. Introduction

Any real-valued non-stationary signal f of finite energy, that is, f is in the space $L^2(\mathbb{R})$ of square integrable functions on the set $\mathbb{R}$ of real numbers, may be represented as an *amplitude-phase modulation* with a time-varying amplitude ρ and a time-varying phase ϕ where phase ϕ is, in general, *nonlinear* [14]. Specifically, the value of f at $t\in\mathbb{R}$ may be represented as

$$f(t)=\rho(t)\cos\phi(t). \tag{1.1}$$

Unfortunately, this type of representation is not unique because the modulation is obtained through a complex signal that can have various choices of the imaginary part. However, one can determine a unique such factorization (1.1) by using the approach of analytic signals [10]. Indeed, let $\mathcal{A}(f)$ be the *analytic signal associated with* f with the characteristic property

$$(\mathcal{A}(f))^{\wedge}(\omega)=\begin{cases}2\hat{f}(\omega) & \text{if } \omega\geq 0\\ 0 & \text{if } \omega<0,\end{cases} \tag{1.2}$$

where for any signal $g\in L^2(\mathbb{R})$, $\hat{g}=\mathcal{F}g$ is the Fourier transform of g defined at $\xi\in\mathbb{R}$ by the equation

$$\hat{g}(\xi)=(\mathcal{F}g)(\xi):=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} g(t)e^{-i\xi t}\,dt. \tag{1.3}$$

Eq. (1.2) is equivalent to for $t\in\mathbb{R}$,

$$\mathcal{A}(f)(t)=f(t)+i\mathcal{H}f(t),$$

where the operator $\mathcal{H}:L^2(\mathbb{R})\to L^2(\mathbb{R})$ stands for the Hilbert transform, and for $f\in L^2(\mathbb{R})$, $\mathcal{H}f$ at $t\in\mathbb{R}$ is defined through the principal value integral

$$\mathcal{H}f(t):=\text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}}\frac{f(x)}{t-x}\,dx=\lim_{\epsilon\to 0}\int_{|x-t|>\epsilon}\frac{f(x)}{t-x}\,dx.$$

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The value $\mathcal{A}(f)(t)$ at $t\in\mathbb{R}$ is complex which can be written into the *quadrature form*

$$\mathcal{A}(f)(t) = \rho(t)e^{i\phi(t)}.$$

Under the conditions that the derivative value $\phi'(t)$ is non-negative for all $t\in\mathbb{R}$, the quantities $\rho(t)$ and $\phi'(t)$ are called the *instantaneous amplitude* and *instantaneous frequency* at $t\in\mathbb{R}$, of the real signal *f*, respectively. The corresponding modulation (1.1) is then called the *canonical amplitude-phase modulation,* or *canonical modulation* for short [8]. The signal *f* with such defined non-negative instantaneous frequencies is thus called a *mono-component*. A large body of literature addresses this problem, see for example, [1,2,16,17,9,21].

The notions of instantaneous amplitude and frequency are fundamental in many applications involving modulations of signals that appear especially in communications or information processing. Thus constructing the canonical pair (ρ, ϕ) of the instantaneous amplitude and phase is important in the theories of analytic signals and in order to facilitate the modulation and demodulation techniques such as processing speech signals [15] or signals in electrical and radio engineering [11]. It is equivalent to the problem of seeking the function pair (ρ, ϕ) such that for $t\in\mathbb{R}$, the following equation holds true:

$$\mathcal{H}(\rho(\cdot)\ \cos\ \phi(\cdot))(t) = \rho(t)\ \sin\ \phi(t). \tag{1.4}$$

We remark that (1.4) can be apparently considered as a special case of the *Bedrosian identity*

$$\mathcal{H}(fg) = f\mathcal{H}(g).$$

In [1], the author proved that, if both $f,\ g$ belong to $L^2(\mathbb{R})$, f is of lower frequency, g is of higher frequency and $f,\ g$ have no overlapping frequency, then $\mathcal{H}(fg) = f\mathcal{H}(g)$. This classic result of Bedrosian is not useful for constructing a mono-component. The reason lies in that the requirement of both f and g in $L^2(\mathbb{R})$ is invalid.

Recently, an important phase function that renders mono-components was given in [18]. The phase function is defined through the boundary values of a Blaschke product on a unit disk $\Delta := \{z : z\in\mathbb{C},\ |z|\le 1\}$, where $\mathbb{C}$ indicates the set of complex numbers. Specifically, for $a\in(-1,\ 1)$, the Blaschke product at $z\in\mathbb{C}\setminus\{1/a\}$ is given by

$$B_a(z) = \frac{z-a}{1-az}. \tag{1.5}$$

Subsequently the *non-linear phase* function, denoted by θ_a, is defined at $t\in\mathbb{R}$ by the equation

$$e^{i\theta_a(t)} := B_a(e^{it}). \tag{1.6}$$

If we recall that the periodic Poisson kernel p_a whose value at $t\in\mathbb{R}$ is given by

$$p_a(t) := \frac{1-a^2}{1-2a\ \cos\ t + a^2}, \tag{1.7}$$

then by taking the derivative of both sides of Eq. (1.6), we find that the phase θ_a is an anti-derivative of p_a, and its derivative is always positive, that is,

$$\frac{d}{dt}\theta_a(t) = p_a(t) > 0.$$

We shall in this paper characterize the amplitude function ρ when the phase function ϕ is chosen at $t\in\mathbb{R}$ by

$$\phi(t) = \theta_a(t) = \int_{[0,t]} p_a(x)\ dx$$

such that Eq. (1.4) is satisfied. Our main result indicates that such kind of amplitude can be perfectly reconstructed in terms of a sampling formula using the *generalized sampling function* whose value at $t\in\mathbb{R}$ is given by

$$\mathrm{sinc}_a(t) := \frac{\sin\ \theta_a(t)}{t}. \tag{1.8}$$

In Section 2, we review the construction of the generalized sampling function and discuss some properties pertaining to it. In Section 3, we introduce the concept of Bedrosian subspace of the Hilbert transform and investigate some properties of functions in this space. In Section 4, we make an important observation when a *linear phase* is chosen, the amplitude function must be bandlimited in order to satisfy equation (1.4). In Section 5, we present our main result in Theorem 5.7.

2. Generalized sampling functions

Not very surprisingly the function sinc_a has many properties that are similar to the classic sinc defined at $t\in\mathbb{R}$ by the equation

$$\mathrm{sinc}(t) := \frac{\sin\ t}{t}.$$

Those properties include cardinality, orthogonality, decaying rate, among others. In the special case $a=0$, the function sinc_a reduces to the classic sinc, which will become clear later. Let us first review the approach to obtain an explicit form of sinc_a.

The classic sinc function is fundamentally significant in digital signal processing due to the Shannon sampling theorem [19,20,3]. The Shannon sampling theorem enables to reconstruct a bandlimited signal from shifts of sinc functions weighted by the uniformly spaced samples of that signal. Recently efforts have been made to extend the classic sinc to generalized sampling functions, for example, in [5–7]. Intuitively, the spectrum of the sinc function is just the indicator function of a symmetric interval of finite measure. Hence, authors in [5] are inspired to consider functions with piecewise polynomial spectra to replace the usual sinc function for the purpose of sampling non-bandlimited signals. One kind of generalized sampling functions given in [5], denoted by sinc_a that is related to a constant $a\in(-1,\ 1)$, is defined as the *inverse Fourier transform* of a so-called *symmetric cascade filter*, denoted by H_a. Specifically,

$$\mathrm{sinc}_a := \sqrt{\frac{\pi}{2}}(1+a)\mathcal{F}^{-1}H_a. \tag{2.1}$$

Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{Z}$ be the set of integers, and $\mathbb{Z}_+ := \{0\}\cup\mathbb{N}$. Let X be a subset of $\mathbb{R}$, and for $q\in\mathbb{N}$, we say a function f is in $L^q(X)$ if and only if the $L^q(X)$ norm

$$\|f\|_{q,X} := \left(\int_X |f(t)|^q\ dt\right)^{1/q} < \infty.$$