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Instantaneous frequency and amplitude of orthocomplex modulated signals based on quaternion Fourier transform

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ABSTRACT

The ideas of instantaneous amplitude and phase are well understood for signals with real-valued samples, based on the analytic signal which is a complex signal with one-sided Fourier transform. We explore the extension of these ideas to signals with complex-valued samples, using a quaternion-valued equivalent of the analytic signal obtained from a one-sided quaternion Fourier transform which we refer to as the *hypercomplex representation* of the complex signal. We discuss its derivation and properties and show how to obtain a complex envelope and a real phase from it.

A classical result in the case of real signals is that an amplitude modulated signal may be decomposed into its envelope and carrier using the analytic signal provided that the modulating signal has frequency content not overlapping with that of the carrier. We show that this idea extends to the complex case, provided that the complex signal modulates an *orthonormal* complex exponential. Examples are presented to demonstrate these concepts.

1. Introduction

The instantaneous amplitude and phase of a real-valued signal have been understood since 1948 from the work of Ville [25] and Gabor [13] based on the *analytic signal*. A critical discussion of instantaneous amplitude and phase was given by Picinbono in 1997, particularly with reference to amplitude and phase modulation [16, Section V]. The analytic signal has the interesting property that its modulus $|a(t)|$ is an envelope of the signal $f(t)$. The envelope is also known as the *instantaneous amplitude*. Thus if $f(t)$ is an amplitude-modulated sinusoid, the envelope $|a(t)|$, *subject to certain conditions on the frequency content*, is the modulating signal. The argument of the analytic signal, $\angle a(t)$, is known as the *instantaneous phase* and its derivative is known as the *instantaneous frequency*.

In this paper we explore the extension of the above ideas to a certain class of signal with complex-valued samples. The concept of the instantaneous frequency of a complex signal is likely to be open to multiple interpretations, and therefore our results are unlikely to be definitive, but a key idea in what we present is the evolution of the complex signal in a 3D space with real, imaginary, and time axes. The angular rate of evolution of the signal, is, we believe, a possible useful concept of instantaneous frequency. This concept is a natural extension of the classical case when the complex signal is orthocomplex modulated.

We show that if a quaternion Fourier transform is computed from a complex-valued signal, and negative frequencies in the frequency domain representation are suppressed, the inverse quaternion Fourier transform gives a generalisation of the analytic signal, which we refer to in this paper as a *hypercomplex representation* of the complex

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signal. Just as the classical analytic signal represents a real-valued signal by a complex signal (that is with a pair of real signals), the hypercomplex representation presented here represents a complex-valued signal by a quaternion signal (that is with a pair of complex signals, based on the Cayley–Dickson form of a quaternion).

We consider signals with complex-valued samples with some restrictions. We exclude the special case of an *analytic signal* (where the real and imaginary parts are orthogonal) because it does not possess an interesting hypercomplex representation: the two additional quaternion components of the hypercomplex representation are simply copies of the two original components. More importantly, we consider complex signals in which the real and imaginary parts have, in some sense, similar frequency content. The reason for this is that the concept of instantaneous frequency is not likely to be meaningful if the real and imaginary parts have frequency content which is in widely separated bands. Consider, for example, a signal in which the real part has a single low frequency, while the imaginary part has a single high frequency (significantly higher than the frequency of the real part). Clearly the complex signal is oscillating in two independent directions and attempting to define an overall instantaneous frequency is not likely to be useful. In contrast, if a complex signal has oscillations in two or more directions with similar frequencies, the signal traces out a path in the complex plane over time that may have a plausible instantaneous frequency.

In the literature it has been customary to handle complex signals by using an augmented representation consisting of a vector containing the signal and its conjugate [24]. There have been several different approaches to modelling bivariate signals (equivalent to complex signals) which we mention here in order to set our own work in context, even though the aim of the other work is to model specific types of signal rather than to extend the classical theory of the analytic signal, as we do in this paper. Lilly and Olhede [15] have published a paper on bivariate analytic signal concepts. Their approach is linked to a specific signal model, the *modulated elliptical signal*, which they illustrate with the example of a drifting oceanographic float, and they utilise pairs of analytic signals derived from the pair of real signals representing the time-varying positional coordinates of a drifting float. In this paper a different approach, using hypercomplex representation, is considered because it is more suited to the analysis of modulation, and also because we have at our disposal the existing tools of quaternion Fourier transforms, which provide a natural fit with modulation of a complex signal by a complex signal, for reasons which should be evident later in the paper. Note that, in essence, the approach proposed in this paper is inspired by the work of Bülow [2,3] and Labunets [18] who considered Clifford Fourier transforms (quaternion in the case of Bülow) to process multidimensional signals. While Bülow considered 2D signals (images), here we consider 1D signals with 2D samples (complex valued samples). We are not focusing on any particular application in this paper, rather we propose a new approach to the processing of orthocomplex modulated signals. Schreier and Scharf's book gives examples of applications [24].

Another approach, based on extending the Empirical Mode Decomposition to a bivariate signal, is proposed in [17]. In this paper the authors develop a concept of *envelope* as we do in this paper, although the underlying method is very different, not being based, as our approach is, on a one-sided Fourier transform. A further approach inspired by the Empirical Mode Decomposition, but based on wavelets, is presented in [7].

We explore the extension of the idea of amplitude modulation to the case where the modulating signal is complex, and the 'carrier' is an orthonormal complex exponential, that is a complex exponential which is in a complex plane orthogonal to the complex plane of the modulating signal – this concept is easily realised using quaternions, since quaternion algebra has a four-dimensional basis with three mutually orthogonal imaginary units. Subject to the same restrictions on frequency content as in the real-valued case (due to Bedrosian [1]), we find that amplitude modulation in the time domain corresponds to a frequency shift in the Fourier domain. Further, a complex envelope of the hypercomplex representation can be defined which recovers the *complex* modulating signal. This is the *complex* instantaneous amplitude. Further, the phase of the orthonormal complex exponential may be recovered. This is the instantaneous phase, and its derivative is the instantaneous frequency. In the case of modulation of a complex exponential the instantaneous frequency is that of the complex exponential 'carrier'. Just as the classical complex analytic signal contains both the original real signal (in the real part) and a real orthogonal signal (in the imaginary part), our hypercomplex signal representation contains two complex signals: the original signal and an orthogonal signal.

Note that we do not consider the case of a complex carrier modulated by a complex modulating signal (as opposed to an *orthocomplex* modulating signal). To see why, consider a complex exponential carrier $f(t)$, modulated by a complex modulating signal $g(t)$, yielding a modulated signal $m(t) = f(t)g(t)$. Now rotate the carrier by an angle ψ in the complex plane by scaling it by a constant complex exponential with angle ψ, yielding a modified modulated signal: $m'(t) = [f(t)\exp(i\psi)]\,g(t)$. Clearly, this modified modulated signal is indistinguishable from $m'(t) = f(t)[g(t)\exp(i\psi)]$ in which the complex modulating signal $g(t)$ is instead rotated. If we allowed ψ to vary with time, then we could fix either $f(t)$ or $g(t)$ to be real, and assign the time-varying angle to the other signal. Without the concept of orthocomplex modulation, there is an ambiguity about the arguments of the modulating signal and the carrier. This may be seen as a generalisation of the amplitude and phase modulation ambiguity problem [6, Chapter 1], namely, for a real signal $s(t) = A(t)\cos(\phi(t))$, where $A(t)$ is the modulated amplitude and $\phi(t)$ the modulated phase, there are an infinite number of pairs $(A(t), \phi(t))$ which reconstruct $s(t)$.

The sequence of topics in the rest of the paper is as follows. Section 2 presents the quaternion algebra and quaternion Fourier transform concepts used to construct the hypercomplex representation of a complex signal using a one-sided quaternion spectrum. Section 3 discusses the hypercomplex representation of a complex