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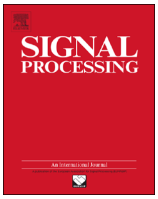

Fractional Hilbert transform extensions and associated analytic signal construction

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ABSTRACT

The analytic signal (AS) was proposed by Gabor as a complex signal corresponding to a given real signal. The AS has a one-sided spectrum and gives rise to meaningful spectral averages. The Hilbert transform (HT) is a key component in Gabor's AS construction. We generalize the construction methodology by employing the fractional Hilbert transform (FrHT), without going through the standard fractional Fourier transform (FrFT) route. We discuss some properties of the fractional Hilbert operator and show how decomposition of the operator in terms of the identity and the standard Hilbert operators enables the construction of a family of analytic signals. We show that these analytic signals also satisfy Bedrosian-type properties and that their time–frequency localization properties are unaltered. We also propose a generalized-phase AS (GPAS) using a generalized-phase Hilbert transform (GPHT). We show that the GPHT shares many properties of the FrHT, in particular, selective highlighting of singularities, and a connection with Lie groups. We also investigate the duality between analyticity and causality concepts to arrive at a representation of causal signals in terms of the FrHT and GPHT. On the application front, we develop a secure multi-key single-sideband (SSB) modulation scheme and analyze its performance in noise and sensitivity to security key perturbations.

1. Introduction

The analytic signal formalism proposed by Gabor [1] is a methodology for constructing a unique complex signal associated with a given real signal. The complex signal is derived by adding the real signal in quadrature with its Hilbert transform. As a consequence of this construction, the spectrum vanishes for negative frequencies. The amplitude-modulation (AM) and phase-modulation (PM) of an AM–frequency-modulated (AM–FM) signal are defined as the modulus and phase of the AS, respectively [2–5]. The AS allows for meaningful computations of the mean frequency and bandwidth, which are crucial in computing the time-bandwidth product and in determining the time–frequency localization properties governed by Heisenberg's uncertainty principle. The analytic nature of the signal can be retained while allowing for specific properties on the AM, FM, or PM. For example, to have a nonnegative FM, finite Blaschke products (which are essentially time-domain duals of allpass filters that signal processing community is more familiar with) have been considered by Picinbono [6], Kumaresan–Rao [7], and Doroslovački [8], which were further extended to an infinite product by Dang and Qian [9], who provided a systematic study of minimum-phase and allpass factorization of analytic signals. Specific results in the context of analytic representations with bandlimited AM have been derived by Xia et al. [10] and Deng and Qian [11]. Another class of generalizations concerning the analytic signal representation are given in terms of employing FrFT, which are reviewed in Section 1.1.

In this paper, we develop a generalization of Gabor's construction using the FrHT without using the FrFT. We

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also propose another generalization of our method, which relies on a functional extension of the HT.

1.1. Review of related work

One of the early works in connection with FrHTs is that of Lohmann et al. [12], who proposed two fractional generalizations of the classical HT. They proposed optical implementations of the FrHT using spatial filtering. One of the generalizations is based on a modification of the spatial filter using a fractional parameter, and the other one is based on the FrFT [13,14]. They also provided a unified definition resulting from a combination of the spatial filter and FrFT approaches. Davis et al. [15] employed the FrHT of Lohmann et al. for edge detection and showed that as the fractional order is varied, one can obtain different qualities of edge enhancement and also achieve selective highlighting of rising or falling edges. Pei and Yeh [16] also deployed the FrHT for similar applications based on a discrete counterpart of the FrHT. Zayed [17] took the AS formalism associated with the standard Fourier transform (FT) and provided a counterpart of it for the FrFT. This procedure is anchored on the FrFT and requires a generalization of the HT, which is obtained by multiplying the time-domain function with a chirp, passing the product through the standard HT, and then dechirping the output to obtain the corresponding AS. Tseng and Pei [18] considered optimized design strategies for finite impulse response (FIR) designs and infinite impulse response (IIR) models of the discrete-time FrHT, and proposed a novel secure SSB communication application using the angle of the FrHT as a secure key. An interesting aspect of their design is that since the FrHT is an allpass filter, much like the HT, rational allpass filter forms are used to approximate the FrHT. Pei and Wang proposed the design of discrete FrHT using maximally flat FIR filters whose impulse responses were obtained in simple analytical form, and devised efficient hardware realizations for both odd and even order filters [19]. Cusmariu proposed three extensions of Gabor's AS using the FrHT, called fractional analytic signals, which reduce to Gabor's AS when the FrHT angle is set to $\pi/2$ [20]. The first fractional AS definition is based on a rotation of Gabor's AS, the second is obtained by using the FrHT as the imaginary part, and the third is obtained as a weighted linear combination of the signal and its standard HT. In particular, it was shown in this work that the FrHT has the semi-group property, unlike the GHT proposed by Zayed. However, fractional analytic signals do not always have a one-sided spectrum. Tao et al. [21] pursued a similar philosophy as that of Zayed and proposed an analytic counterpart associated with the FrFT in conjunction with the FrHT, using time-domain chirping, standard HT, and dechirping. The fundamental issue addressed by Tao et al. is that since the FrFT does not exhibit conjugate symmetry, suppressing the negative part of the frequency spectrum requires some care; otherwise, one may not be able to recover the real signal from the analytic counterpart [21]. On the application front, they proposed a SSB modulation scheme, which uses the angle of the FrFT and the phase of the FrHT as secret keys for demodulation. Fu and Li proposed a generalization of the AS through the linear canonical transform (LCT) domain [22]. Their formalism is based on the parameterized HT, which includes the FrFT-based HT [17] as a special case, and the LCT-based generalized AS is obtained by taking the inverse-LCT after suppressing its negative frequency spectrum. A generalized Bedrosian theorem associated with the LCT was also proposed by Fu and Li. Guanlei et al. [23] proposed different definitions of two-dimensional HT in the LCT domain and discussed different properties, with particular emphasis on the relation between the time-domain and transformed domain mappings. Sarkar et al. proposed a generalization of the HT based on the generalization of decay of the impulse response of the standard HT kernel and applied it for symbolic time series analysis of noise-corrupted dynamical systems [24]. They also proposed an AS counterpart using the generalized HT, which results in functions with one-sided spectra only for certain values of the decay parameter. An all-optical implementation of the FrHT using a phase-shifted fiber Bragg grating was proposed by Ge et al. [25], capable of operating on input waveforms with bandwidths up to hundreds of Gigahertz. Chaudhury et al. [26] deployed the FrHT in the representation of the dual-tree complex wavelet transforms based on the shifting action of the FrHT operator. They also introduced a generalization of the Bedrosian identity for the FrHT, and extended the concepts to the multidimensional setting by proposing a directional FrHT.

1.2. This paper

In this paper, we address the aspect of AS construction starting with the FrHT, but by employing the standard FT and without taking the FrFT route. In fact, it turns out that the properties of the spectra associated with FrHT operators and the approach presented in this paper do not necessitate the use of FrFT. We recall some properties of the FrHT vis-a-vis those of the HT. We show how our framework leads to the generation of a family of analytic signals, one corresponding to every value of the phase parameter of the FrHT. We show that the FrHT approach for AS construction enables a natural geometric interpretation. We carry out the analytical developments and construction of a generalization of the AS in the continuous-domain, which we refer to as the $\phi-$AS, for which we favor a Hilbert space formalism and work with suitably defined operators. We also seek to analyze some specific properties of the $\phi-$AS, namely, AM/FM separation, operator invariances, factorization properties, energy preservation, temporal and spectral moments, and the associated Heisenberg uncertainty principle. We also propose a new member to the family of Hilbert transforms, called the generalized-phase Hilbert transform (GPHT), which is based on a functional generalization of the FrHT phase. We show that the properties of the FrHT, in particular, the enhancement and selective highlighting of edges, are preserved in the process of generalization. A connection between Lie groups and the GPHT is also presented. The GPHT is then used to construct a generalized $\phi-$AS construction, called the generalized-phase analytic signal. We show that the GPAS is related to Gabor's AS by a linear filtering operation. By analyzing the link between causality and analyticity, we show how causal signals can be characterized using the FrHT/GPHT. As an application of the developed concepts, we propose a multi-key secure