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Fast communication

ℓ_p-norm based iterative adaptive approach for robust spectral analysis

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A R T I C L E I N F O

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A B S T R A C T

Recently, the iterative adaptive approach (IAA) has been shown to be an effective nonparametric methodology for high-resolution spectral analysis. Its main idea is to reformulate the nonlinear frequency estimation problem as a linear model whose parameters are updated iteratively according to weighted least squares. Since the derivation of the IAA is based on ℓ_2-norm, it cannot work properly in heavy-tailed noise environment. In this paper, a generalized version of IAA is derived to provide accurate spectral estimation in the presence of impulsive noise, which replaces the ℓ_2-norm by the ℓ_p-norm where $1<p<2$. Simulation results are included to demonstrate the outlier resistance performance of the proposed algorithm.

1. Introduction

Spectral analysis has been an important topic in science and engineering because many real-world signals are well described by the sinusoidal model. Basically, the frequency components of the observed data can be obtained by means of either parametric or nonparametric techniques [1]. In the parametric approach, the signal is assumed to satisfy a generating model with known functional form, which allows the derivation of the optimal spectral estimators. However, the performance of these methods degrades when there is a mismatch between the assumed and actual signal models. On the other hand, no assumptions are made about the data in the nonparametric approach. A conventional representative is the periodogram, which is based on the Fourier transform, but its resolution is fundamentally limited by the available observation length. Recently, the iterative adaptive approach (IAA) [2,3] provides a breakthrough in the nonparametric methodology because of its very high accuracy and resolution. This method can be interpreted as reformulating the nonlinear spectral estimation problem as a linear model whose coefficients, representing amplitudes at different frequencies on a fine grid, are updated iteratively according to weighted least squares (WLS). Since its development is based on ℓ_2-norm optimization, the IAA is not robust to heavy-tailed noise, which occurs in many fields such as wireless communications, radar and sonar [4]. Typical models for impulsive noise include $\alpha-$stable noise [5], Gaussian mixture model (GMM) [4], and generalized Gaussian distribution (GGD) [6].

Existing methods for robust spectral analysis include [7–11]. With the use of Huber's minimax robust statistics [12], Katkovnik [7] has developed a maximum-likelihood (ML) type *M*-periodogram for spectral analysis in heavy-tailed noise. Its extension to time–frequency analysis for nonstationary signals is provided in [8]. Linear combination of order statistics, which is also called *L*-estimation, has been studied in [9] for robust transforms and time–frequency representations. In [11], it is proved that the least absolute deviation (LAD) estimator provides the ML performance in the presence of Laplacian white noise. Recently, the concept of LAD is generalized to ℓ_p-norm minimization where $1\le p<2$ in [11]. In this paper, by

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utilizing ℓ_p−norm minimization and IAA, we derive an outlier-resistant version of IAA, which is referred to as ℓ_p−IAA, for impulsive noise environment.

The rest of this paper is organized as follows. The development of ℓ_p−IAA is presented in Section 2. We can see that the ℓ_p−IAA generalizes [2] as it reduces to the original IAA by setting $p=2$. Computer simulations in Section 3 show the effectiveness of the proposed scheme as well as its superiority over the IAA in the presence of both α−stable and GMM noises. Finally, conclusions are drawn in Section 4.

2. Proposed method

Without loss of generality, the observed signal sequence is modeled as

$$y_n = \sum_{l=1}^{L} \gamma_l e^{j\varpi_l n} + q_n, \quad n=0,\ldots,N-1 \tag{1}$$

where γ_l and $\varpi_l \in (0,2\pi)$ are the complex amplitude and frequency of the lth tone, respectively, L is the number of sinusoids, and q_n is an isotropic independent and identically distributed (IID) heavy-tailed noise. It is assumed that the phases of $\{\gamma_l\}$ are independently and uniformly distributed in $[0,2\pi]$ [1] and L is unknown. Our task is to find $\{\varpi_l\}$ from $\{y_n\}$.

Although frequency estimation corresponds to a nonlinear model, (1) can be reformulated as the following linear model [2]:

$$\mathbf{y} = \mathbf{A}\mathbf{x} \tag{2}$$

where $\mathbf{y} = [y_0 \ \cdots \ y_{N-1}]^T$ with ^{T} denotes transpose, $\mathbf{A} = [\mathbf{a}(\omega_0) \ \cdots \ \mathbf{a}(\omega_{K-1})]$ with $\mathbf{a}(\omega_k) = [a_0(\omega_k) \ \cdots \ a_{N-1}(\omega_k)]^T$, $a_n(\omega_k) = e^{jn\omega_k}$, $\omega_k = 2\pi k/K$, $k=0,\ldots,K-1$, and $\mathbf{x} = [x_0 \ \cdots \ x_{K-1}]^T$. Note that the noise components $\{q_n\}$ are absorbed in $\mathbf{x}$. Here, the admissible frequency interval $[0,2\pi]$ is divided into K uniform grid points $\{\omega_k\}$ while x_k and $\mathbf{a}(\omega_k)$ are the amplitude and frequency-vector associated with ω_k, respectively. Assuming that K is chosen sufficiently large and in the absence of noise, we have

$$x_k = \begin{cases} \gamma_l, & \omega_k = \varpi_l, \quad l=1,\ldots,L \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

To achieve outlier resistance, the proposed ℓ_p-IAA replaces the ℓ_2−norm in [2] by ℓ_p−norm where $1<p<2$, and the conceptual estimate of x_k, denoted by $\hat{x}_k$, is

$$\hat{x}_k = \arg\min_{x_k} \|\mathbf{y} - \mathbf{a}(\omega_k)x_k\|^p_{\mathbf{Q}_k^{-1}} \tag{4}$$

where

$$\begin{aligned} \|\mathbf{y} - \mathbf{a}(\omega_k)x_k\|^p_{\mathbf{Q}_k^{-1}} &= \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} [\mathbf{Q}_k^{-1}]_{i,j}(y_i - a_i(\omega_k)x_k)^* \\ &\quad |y_j - a_j(\omega_k)x_k|^{p-2}(y_j - a_j(\omega_k)x_k) \\ &= (\mathbf{y} - \mathbf{a}(\omega_k)x_k)^H \mathbf{Q}_k^{-1}\mathbf{W}_k(\mathbf{y} - \mathbf{a}(\omega_k)x_k). \end{aligned} \tag{5}$$

Here, $^{-1}$, * and ^{H} denote matrix inverse, conjugate, and conjugate transpose, respectively, $\mathbf{Q}_k^{-1}$ is the weighting matrix with (i,j) entry $[\mathbf{Q}_k^{-1}]_{i,j}$, and $\mathbf{W}_k = \mathrm{diag}(|y_0 - a_0(\omega_k)x_k|^{p-2} \ \cdots \ |y_{N-1} - a_{N-1}(\omega_k)x_k|^{p-2})$ is a diagonal matrix. The $\mathbf{Q}_k$ has the form [13]

$$\mathbf{Q}_k = E\{(\mathbf{y} - \mathbf{a}(\omega_k)x_k)(\mathbf{y} - \mathbf{a}(\omega_k)x_k)^H \mathbf{W}_k\} \tag{6}$$

where E denotes the expectation operator.

Based on iteratively reweighted least squares, we solve (4) in an iterative manner and the estimate at the $(l+1)$th iteration is computed as

$$\begin{aligned} \hat{x}_k^{(l+1)} &= \arg\min_{x_k}\{(\mathbf{y} - \mathbf{a}(\omega_k)x_k)^H (\mathbf{Q}_k^{(l)})^{-1}\mathbf{W}_k^{(l)}(\mathbf{y} - \mathbf{a}(\omega_k)x_k)\} \\ &= \frac{\mathbf{a}^H(\omega_k)(\mathbf{Q}_k^{(l)})^{-1}\mathbf{W}_k^{(l)}\mathbf{y}}{\mathbf{a}^H(\omega_k)(\mathbf{Q}_k^{(l)})^{-1}\mathbf{W}_k^{(l)}\mathbf{a}(\omega_k)}, \quad k=0,\ldots,K-1 \end{aligned} \tag{7}$$

where $(\mathbf{Q}_k^{(l)})^{-1}\mathbf{W}_k^{(l)}$ is the reweighted matrix which is characterized by $\{\hat{x}_k^{(l)}\}$. In the Appendix, we have shown that (7) can be further simplified as

$$\hat{x}_k^{(l+1)} = \frac{\mathbf{a}^H(\omega_k)(\mathbf{R}^{(l)})^{-1}\mathbf{W}_k^{(l)}\mathbf{y}}{\mathbf{a}^H(\omega_k)(\mathbf{R}^{(l)})^{-1}\mathbf{W}_k^{(l)}\mathbf{a}(\omega_k)}, \quad k=0,\ldots,K-1 \tag{8}$$

where

$$\mathbf{R}^{(l)} = \mathbf{A}\,\mathrm{diag}(|\hat{x}_0^{(l)}|^p \ \cdots \ |\hat{x}_{K-1}^{(l)}|^p)\mathbf{A}^H \tag{9}$$

is the robust covariance matrix constructed from $\{\hat{x}_k^{(l)}\}$ and

$$\mathbf{W}_k^{(l)} = \mathrm{diag}(|y_0 - a_0(\omega_k)\hat{x}_k^{(l)}|^{p-2} \ \cdots \ |y_{N-1} - a_{N-1}(\omega_k)\hat{x}_k^{(l)}|^{p-2}). \tag{10}$$

The steps of the proposed algorithm are summarized in Table 1. It is worth noting that (8) reduces to the standard IAA when $p=2$. That is to say, the ℓ_p−IAA is a generalized version of [2].

The computational complexities of the proposed method as well as standard IAA and its fast implementation [14,15] are now examined. At each iteration, the numbers of multiplications required are $2KN^2 + KN + 2N^3$, $2KN^2 + KN + N^3$ and $N^2 + 24N\log_2(2N) + 3K\log_2(K)$, respectively. The additional computational cost of N^3 in ℓ_p−IAA over the conventional scheme is due to the multiplication of $(\mathbf{R}^{(l)})^{-1}$ and $\mathbf{W}_k^{(l)}$. That is to say, when the number of iterations is kept identical in all schemes, the fast implementation is the most computationally efficient. Nevertheless, we can follow [14,15] to produce a fast implementation for the ℓ_p−IAA.

3. Simulation results

To evaluate the spectral estimation performance of ℓ_p−IAA, computer simulations have been conducted. All results are based on 100 independent runs and $p=1.2$ is selected. First, we examine the mean square frequency error (MSFE) performance of the proposed algorithm for a single tone. We also include the results of the standard IAA and Cramér–Rao lower bound (CRLB). The stopping criterion in the ℓ_p-IAA and IAA is chosen according to [2]. In the first test, the signal is generated according to (1) where

Table 1
Summary of proposed algorithm.

(i)	Initialize the values of $\{\hat{x}_k^{(0)}\}$ using discrete Fourier transform
(ii)	Compute $\mathbf{R}^{(l)}$ and $\mathbf{W}_k^{(l)}$ using (9) and (10) for $k=0,\ldots,K-1$
(iii)	Update $\hat{x}_k$ using (8) for $k=0,\ldots,K-1$
(iv)	Repeat Steps (ii)–(iii) until a stopping criterion is reached