Signal Processing 89 (2009) 1032–1037

Contents lists available at ScienceDirect

Signal Processing

journal homepage: www.elsevier.com/locate/sigpro

Minimum-phase parts of zero-phase sequences

Corneliu Rusu [a,*], Jaakko Astola [b]

[a] Faculty of Electronics, Telecommunications and Information Technology, Technical University of Cluj-Napoca, Cluj-Napoca, Str. Baritiu 26-28, RO-400027, Romania
[b] Tampere International Center for Signal Processing, Tampere University of Technology, Tampere, Finland

ARTICLE INFO

Article history:
Received 26 May 2008
Received in revised form
8 September 2008
Accepted 4 December 2008
Available online 24 December 2008

Keywords:
Signal reconstruction
Discrete-time Fourier transform
Discrete Fourier transform
Nonuniform discrete Fourier transform
Minimum-phase function

ABSTRACT

For more than a decade it has been empirically known that the causal portion of the inverse Fourier transform of the magnitude spectrum of the speech signal behaves like a minimum-phase signal. Later on, this statement has been shown for an all-pole model. In this paper, we consider related results for both discrete-time Fourier and discrete Fourier transforms of arbitrary sequences. We indicate how the presence of aliasing in circular autocorrelation might be detected. The energy concentration property of zero-phase sequences is discussed.

1. Introduction

In many cases where it is necessary to avoid any phase distortion, one solution is to make the frequency response real and nonnegative, i.e., to have a filter with a zero-phase characteristic [1]. Besides filter design, zero-phase signals appear in applications like phase retrieval or phase-only reconstruction [2] and in speech analysis [3]. In spectral factorization theorems, the zero-phase condition represents a key assumption. Perhaps the most known result on zero-phase sequences is the celebrated Fejér and Riesz theorem [4]:

Theorem 1 (*Fejér and Riesz*). *If* $X(z) = \sum_{n=-M}^{M} x(n)z^{-n}$ *and* $X(e^{j\omega}) \geqslant 0$, *then there is* $Y(z) = \sum_{n=0}^{M} y(n)z^{-n}$ *such that* $X(e^{j\omega}) = |Y(e^{j\omega})|^2$ *and* $Y(z)$ *unique if minimum-phase.*

In this paper, we discuss some properties of minimum-phase and zero-phase sequences. As the properties of such sequences are related to the properties of rational functions and analytic functions in general, as well as those of Fourier series, results that are closely connected appear implicitly in circuits and systems literature. About two decades ago it was shown that if a positive real function belongs to the class of rational functions, then it must be minimum-phase [5]. To best of our knowledge, there is no published result indicating that the Fourier transform of a causal part of an arbitrary zero-phase sequences is a positive real function belonging to the class of rational functions. Recently, it was noticed that the causal portion of the inverse Fourier transform of the magnitude spectrum of the speech signal behaves like a minimum-phase signal [6]. This property has been verified for an all-pole model [7,8]. Here we present related properties that may be of interest from a digital signal processing point of view, but do not appear explicitly in the literature.

In the following we recall few definitions (Section 2). We consider the causal part of an arbitrary zero-phase sequence and show that it is a minimum-phase sequence (Section 3). We also give a direct proof of the fact that the inverse DFT transform of a strictly positive sequence is minimum-phase (Section 4). We also extend this to NDFT (nonuniform discrete Fourier transform) in Section 5.

* Corresponding author. Tel.: +40 2 64 401804; fax: +40 2 64 591340.
E-mail addresses: Corneliu.Rusu@ieee.org (C. Rusu), Jaakko.Astola@tut.fi (J. Astola).

0165-1684/$ - see front matter
doi:10.1016/j.sigpro.2008.12.006

The presence of aliasing in circular autocorrelation and the energy concentration property of zero-phase sequences are discussed in Section 6.

2. Definitions

Definition 2. A sequence is causal if $x(n)=0$ for $n<0$. A sequence is anti-causal if $x(n)=0$ for $n>0$.

Definition 3. A sequence is minimum-phase if all zeros of its z-transform are inside the unit open disk. A sequence is maximum-phase if all zeros of its z-transform are outside the unit closed disk.

Definition 4. A sequence is absolute summable or stable sequence if the region of convergence of its z-transform includes the unit circle.

Definition 5. A complex valued function $X(z)$ is positive real if

(1) $z \in \mathbb{R} \Rightarrow X(z) \in \mathbb{R}$;
(2) $|z| \geqslant 1 \Rightarrow \mathrm{Re}\{X(z)\} \geqslant 0$.

Definition 6. A sequence is zero-phase sequence if its Fourier transform is a nonnegative function.

Sometimes a zero-phase sequence is said to be a positive sequence [9]; in this paper, we shall refer to this as zero-phase sequence, to distinguish from positive real function.

3. Zeros location of causal part of zero-phase aperiodic sequences

Let $x(n)$ be zero-phase sequence with no zeros or poles on the unit circle. Then its DTFT (discrete-time Fourier transform)

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

satisfies

$$|X(e^{j\omega})| > 0, \quad \arg X(e^{j\omega}) = 0.$$

In this case by using inverse DTFT, the sequence can be computed as follows:

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|e^{j\omega n}\,d\omega.$$

The z-transform of causal portion of $x(n)$ is the one-sided z-transform:

$$X^+(z) = \sum_{n=0}^{\infty} x(n)z^{-n} = \sum_{n=0}^{\infty}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|e^{j\omega n}\,d\omega\right]z^{-n}. \qquad (1)$$

Our goal is to show that the causal portion of $x(n)$ is a minimum-phase sequence.

Theorem 7. *Let $x(n)$ be a stable sequence. If $|X(e^{j\omega})| > 0$ and $\arg X(e^{j\omega}) = 0$ for all $\omega \in \mathbb{R}$, then its causal portion is a minimum-phase sequence.*

Proof. First we establish that for any $|z|>1$, we have $\mathrm{Re}\{X^+(z)\} > 0$.

If $|z|>1$, then the sequence $e^{j\omega n}z^{-n}$ is absolute summable. In such situation we can interchange the order of summation and integration in (1). We get

$$X^+(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\sum_{n=0}^{\infty} e^{j\omega n}z^{-n}\right]|X(e^{j\omega})|\,d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|X(e^{j\omega})|}{1-e^{j\omega}z^{-1}}\,d\omega.$$

Now, for $z = r\,e^{j\theta}$, $r>1$

$$X^+(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|X(e^{j\omega})|\,d\omega}{1-r^{-1}\,e^{j(\omega-\theta)}},$$

$$\mathrm{Re}\left\{\frac{|X(e^{j\omega})|}{1-r^{-1}\,e^{j(\omega-\theta)}}\right\} = \frac{|X(e^{j\omega})|[1-r^{-1}\cos(\omega-\theta)]}{|1-r^{-1}e^{j(\omega-\theta)}|^2} \geqslant \frac{|X(e^{j\omega})|(1-r^{-1})}{(1+r^{-1})^2}.$$

Thus

$$\mathrm{Re}\{X^+(z)\} \geqslant \frac{1-r^{-1}}{2\pi(1+r^{-1})^2}\int_{-\pi}^{\pi}|X(e^{j\omega})|\,d\omega > 0.$$

It follows that for any $|z|>1$, we have $X^+(z) \neq 0$.

To show that $X^+(e^{j\omega_0}) \neq 0$, first notice that

$$x(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\,d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}|X(e^{j\omega})|\,d\omega > 0. \qquad (2)$$

Assume now that $X^+(e^{j\omega_0}) = 0$. Then

$$\begin{aligned} X(e^{j\omega_0}) &= X^+(e^{j\omega_0}) + \sum_{n=-\infty}^{-1} x(n)e^{-j\omega_0 n} = X^+(e^{j\omega_0}) - x(0) \\ &\quad + \sum_{n=-\infty}^{0} x(n)e^{-j\omega_0 n} \\ &= X^+(e^{j\omega_0}) - x(0) + \sum_{n=0}^{\infty} x(-n)e^{j\omega_0 n} = X^+(e^{j\omega_0}) - x(0) \\ &\quad + \sum_{n=0}^{\infty} x(n)^* e^{j\omega_0 n} \\ &= X^+(e^{j\omega_0}) - x(0) + \left[\sum_{n=0}^{\infty} x(n)e^{-j\omega_0 n}\right]^* \\ &= X^+(e^{j\omega_0}) - x(0) + [X^+(e^{j\omega_0})]^* = -x(0). \end{aligned}$$

This is a contradiction since $X(e^{j\omega})$ is positive for all $\omega \in [-\pi,\pi]$. □

Example 8. Let $X(z) = z + \frac{5}{2} + z^{-1}$, then $X(e^{j\omega}) = \frac{1}{2} + 4\cos^2\omega/2 > 0$.

The causal portion of $X(z)$ is $X^+(z) = \frac{5}{2} + z^{-1}$ and its zero is $z_1 = -\frac{2}{5}$.

According to Fejér–Riesz theorem, there is $Y(z)$ such that $X(e^{j\omega}) = |Y(e^{j\omega})|^2$. Indeed, for $Y_1(z) = \sqrt{2}(1+1/2z^{-1})$ and $Y_2(z) = \sqrt{2}/2(1+2z^{-1})$ we have $X(e^{j\omega}) = |Y_1(e^{j\omega})|^2 = |Y_2(e^{j\omega})|^2$. Moreover, $Y_1(z)$ is minimum-phase function, and $Y_2(z)$ is maximum-phase function.

Corollary 9. *For any sequence having no zeros or poles on unit circle, the causal portion of autocorrelation is minimum-phase sequence.*

Example 10. The causal portion of a symmetric sequence may be minimum-phase, but the sequence is not always zero-phase, e.g., $X(z) = 0.7z + 1 + 0.7z^{-1}$.