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A new boundary meshfree method for potential problems

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ABSTRACT

This work presents a new boundary meshfree method, named the average source method (ASM), for solving two-dimensional potential problems. The method is based on combining a 'completely' regularized boundary integral equation (CRBIE) with indirect unknowns developed in this paper, removing the singularity computation, and an average source technique (AST). In this approach there are two critical developments. One is the presentation of a new removal singularity technique that results in the CR-BIE, and therefore all diagonal coefficients of influence matrices can be evaluated analytically by the off-diagonal ones, unlike some existing meshless boundary approaches that determine diagonal coefficients from the fundamental solution by using a known solution, thereby doubling the solution procedure. The other is to introduce an AST, by which the distributed source on a segment/cell can be reduced to the concentrated point source and therefore the boundary integrals in the CRBIE are not necessary. Hence, in the ASM only boundary nodes are required for computation without involving any integration and element notion. Several benchmark test examples are presented to demonstrate the accuracy, convergence, efficiency and robustness of this new meshfree boundary-node methodology.

1. Introduction

As is known to all, the finite element method (FEM) and boundary element method (BEM) have been the dominant numerical engines for science and engineering applications [1–8,36]. However, they require resorting to an element frame for interpolants of primary variables and the 'energy' integration and thus, depend on the generation of meshes, which can be arduous, time-consuming and even subjected to pitfalls, especially for complex geometry domains. These difficulties can be sidestepped via the so-called meshless/meshfree techniques, which have drawn growing attention during the past decades and achieved outstanding progress in solving a wide class of boundary value problems [8–35].

Among the aforementioned studies, the meshless boundary methods have achieved remarkable progress and can be roughly sorted into two categories: the MFS-based type and the BIE-based type. The former is based on the concept of the method of fundamental solution (MFS), including, but are not limited to, the MFS [12–14], the boundary knot method (BKM) [15–16], the boundary collocation method (BCM) [17], the modified MFS (MMFS) [18], the boundary distributed source method (BDSM) [19–20], the regularized meshless method (RMM) [21–22], and the singular boundary method (SBM) [23–25]. The MFS, BKM and BCM generally lead to the ill-conditioned system. The MMFS and the BDSM need to compute some particular integrals to determine the diagonal terms. The RMM uses double layer kernel to express the potential to easily remove the singularity, but the bewildering hyper singularity issue has to be faced when the boundary flux solutions are required. The SBM uses the null-field integral identity firstly to obtain the diagonal terms from the derivative of the fundamental solution, and then it applies a known solution to determine the diagonal terms from the fundamental solution [23–25]. Therefore, as stated in Ref. [19], this approach amounts to solving the problem twice. In addition, the theoretical analysis of this approach is not rigorous, since it uses a false integral identity [23–25]: $\int_{\Gamma} \frac{\partial u^{*c}(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}(\boldsymbol{y})} d\Gamma(\boldsymbol{y}) = 0, \boldsymbol{x} \in \Gamma$ with $u{*}^{c}(\boldsymbol{x}, \boldsymbol{y})$ beings the fundamental solution of the exterior problems. The latter category [8–11,26–31] is based on combining BIEs with meshless shape functions constructed usually by using the moving least-square (MLS) approximation. It is mainly represented by the boundary node method (BNM) [26] and its variants [27–31]. These methods exploit the merits of both the BIE in dimensionality reduction and the MLS in element removal. The essential difference between these methods consists in the construction of meshless shape functions. Anyhow, they still require the calculation of boundary integrals.

Inspired by the pioneering work, this study presents a new meshfree boundary method for 2D potential problems. The method

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is based on combing a CRBIE with direct unknowns developed in this paper, which excludes the computation of both the weakly and strongly singular integrals, with the AST. By using the CRBIE to avoid the singularity of the kernel functions, the major challenge of the coincidence of the source and collocation points vanishes. By introducing the AST into the CRBIE, the distributed source on a segment/cell can be reduced to the concentrated point source and therefore the boundary integrals are no longer required. Since no known solution in the MMFS, RMM and SBM is applied for computing indirectly the diagonal coefficients of influence matrices, the problem can be solved only once with the present approach. Again, unlike the foregoing MLS-based methods [26–31] which is based on introducing MLS-based meshless shape functions constructed elaborately into BIEs and is 'truly meshless' but still involves the calculation of boundary integrals, the present ASM only requires boundary nodes for computation without involving any element or integration notion. Consequently, the ASM is easier-to-implement, much more computationally efficient, and theoretically simpler. Furthermore, in the implementation of the ASM, the real geometry of the domain boundary without approximation can be employed for computation as long as the parametric representation of the domain boundary is given.

As usual, the ASM also requires the discretization of the domain boundary into cells, but as stated in Ref. [28], the using of the cells should not be viewed as a shortcoming of meshless/meshfree schemes if these cells can be generated with ease. Actually, the cells in the ASM are essentially distinguished from the boundary elements in BEM and are employed neither for the purpose of interpolation of the primary variables nor for numerical integration just for computing the Jacobian value at nodes, and also there is no limitation on their shape and size, implying that when some of them are partitioned into smaller cells, their adjacent ones are not affected. In this sense, the ASM should be regarded as a "truly meshless or meshfree" method.

The accuracy, stability, efficiency and widely practical applicability are verified in numerical experiments of the Dirichlet and mixed-type continue or discontinue boundary conditions (BCs) of both interior and exterior problems with simple and complicated boundaries.

2. Regularized BIEs and the ASM

In this paper, we always assume that Ω is a bounded domain in R^2, Ω^c its open complement, and Γ their common boundary.

2.1. Boundary value problem

Consider a two-dimensional potential problem in the domain $\hat{\Omega}$ ($\hat{\Omega}=\Omega$ or Ω^c) governed by the Laplace equation

$$\nabla^2 u(\boldsymbol{x})=0, \quad \boldsymbol{x}=(x_1,x_2)\in\hat{\Omega} \tag{1}$$

with boundary conditions (BCs) [1–2,5–7]

$$u(\boldsymbol{x})=\bar{u}(\boldsymbol{x}), \quad \boldsymbol{x}\in\Gamma_1 \tag{2}$$

$$q(\boldsymbol{x})=\frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}(\boldsymbol{x})}=\bar{q}(\boldsymbol{x}), \quad \boldsymbol{x}\in\Gamma_2 \tag{3}$$

when $\hat{\Omega}=\Omega^c$, in order to guarantee the uniqueness of solution of the exterior problems, the following infinity condition must be supplemented [7,33]

$$|u(\boldsymbol{x})|=O(1), \text{ as } \rho=\sqrt{x_1^2+x_2^2}\to\infty \tag{4}$$

where $\Gamma=\Gamma_1\cup\Gamma_2$ is the boundary of $\hat{\Omega}$ with $\Gamma_1\cap\Gamma_2=\emptyset$; $\bar{u}(\boldsymbol{x})$ and $\bar{q}(\boldsymbol{x})$ are the prescribed boundary functions and $\boldsymbol{n}(\boldsymbol{x})$ is the unit outward normal vector at point $\boldsymbol{x}=(x_1,x_2)\in\Gamma$.

2.2. Regularized indirect boundary integral equations *(IBIEs)*

For potential problems in the domain $\hat{\Omega}$ ($=\Omega$ or Ω^c) bounded by boundary Γ, in the absence of body source, the equivalent regularized IBIEs for the problems (1)–(4) can be expressed as [7,33]

$$\int_\Gamma \phi(\boldsymbol{x})d\Gamma=0 \tag{5}$$

$$u(\boldsymbol{y})=\int_\Gamma \phi(\boldsymbol{x})u^*(\boldsymbol{x},\boldsymbol{y})d\Gamma+C, \quad \boldsymbol{y}\in\Gamma \tag{6}$$

$$\frac{\partial u(\boldsymbol{y})}{\partial \boldsymbol{n_y}}=\hat{k}\phi(\boldsymbol{y})+\int_\Gamma[\phi(\boldsymbol{x})-\phi(\boldsymbol{y})]\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n_y}}d\Gamma +\phi(\boldsymbol{y})\int_\Gamma\left[\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n_y}}+\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n_x}}\right]d\Gamma, \quad \boldsymbol{y}\in\Gamma \tag{7}$$

$$\frac{\partial u(\boldsymbol{y})}{\partial \boldsymbol{t_y}}=\int_\Gamma[\phi(\boldsymbol{x})-\phi(\boldsymbol{y})]\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{t_y}}d\Gamma +\phi(\boldsymbol{y})\int_\Gamma\left[\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{t_y}}+\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{t_x}}\right]d\Gamma, \quad \boldsymbol{y}\in\Gamma \tag{8}$$

For the internal point $\boldsymbol{y}\in\hat{\Omega}$, the integral equations can be written as

$$u(\boldsymbol{y})=\int_\Gamma \phi(\boldsymbol{x})u^*(\boldsymbol{x},\boldsymbol{y})d\Gamma+C, \quad \boldsymbol{y}\in\hat{\Omega} \tag{9}$$

$$\frac{\partial u(\boldsymbol{y})}{\partial y_k}=\int_\Gamma \phi(\boldsymbol{x})\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial y_k}d\Gamma, \quad \boldsymbol{y}\in\hat{\Omega}, k=1,2 \tag{10}$$

In Eqs. (5)–(10), $\boldsymbol{x}=(x_1,x_2)$ and $\boldsymbol{y}=(y_1,y_2)$ are the source and the field points, respectively; $\boldsymbol{t_y}=(t_1(\boldsymbol{y}),t_2(\boldsymbol{y}))$ and $\boldsymbol{n_y}=(n_1(\boldsymbol{y}),n_2(\boldsymbol{y}))$ are the unit tangent and outward normal vectors at $\boldsymbol{y}\in\Gamma=\partial\hat{\Omega}$; $\hat{k}$ is 1 or 0, respectively, for the interior domain Ω and the exterior domain Ω^c; $u*(\boldsymbol{x},\boldsymbol{y})$ denotes the fundamental solution for potential problems expressed as

$$u^*(\boldsymbol{x},\boldsymbol{y})=-\frac{1}{2\pi}\ln|\boldsymbol{x}-\boldsymbol{y}| \tag{11}$$

In order to sidestep the direct computation of the weak singular integral in Eq.(6), based on the following integral identities

$$\int_\Gamma n_i(\boldsymbol{x})u^*(\boldsymbol{x},\boldsymbol{y})d\Gamma=\int_\Gamma (x_i-y_i)\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}}d\Gamma, \boldsymbol{y}\in\hat{\Omega}, i=1,\cdots,d \tag{12}$$

which is readily derived by the Green' second identity, and a limit procedure, i.e.

Lemma [33–35]**.** *Let Γ be a piecewise smooth curve (open or closed), and $\hat{\boldsymbol{x}}$ a point on Γ (perhaps a corner). Suppose $h=|\boldsymbol{y}-\hat{\boldsymbol{x}}|$ and $d=\inf_{\boldsymbol{x}\in\Gamma}|\boldsymbol{y}-\boldsymbol{x}|$. If $\psi(\boldsymbol{x})\in C^{0,\alpha}(\Gamma)$ and $h/d\le K_1$(with constant K_1), then there holds*

$$\lim_{\boldsymbol{y}\to\hat{\boldsymbol{x}}}\int_\Gamma \frac{x_k-y_k}{|\boldsymbol{x}-\boldsymbol{y}|^2}[\psi(\boldsymbol{x})-\psi(\hat{\boldsymbol{x}})]d\Gamma_{\boldsymbol{x}} =\int_\Gamma \frac{x_k-\hat{x}_k}{|\boldsymbol{x}-\hat{\boldsymbol{x}}|^2}[\psi(\boldsymbol{x})-\psi(\hat{\boldsymbol{x}})]d\Gamma_{\boldsymbol{x}} \quad (k=1,2)$$

we develop a new boundary element formulation as follows

$$u(\boldsymbol{y})=\int_\Gamma[\phi(\boldsymbol{x})-\phi(\boldsymbol{y})\boldsymbol{n}(\boldsymbol{y})\cdot\boldsymbol{n}(\boldsymbol{x})]u^*(\boldsymbol{x},\boldsymbol{y})d\Gamma +\phi(\boldsymbol{y})\int_\Gamma \boldsymbol{n}(\boldsymbol{y})\cdot(\boldsymbol{x}-\boldsymbol{y})\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n_x}}d\Gamma+C, \boldsymbol{y}\in\Gamma \tag{13}$$

which is named the 'completely' regularized boundary integral equation, because it excludes the computation of both the weakly and strongly singular integrals.