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A new regularized boundary integral equation for three-dimensional potential gradient field

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ABSTRACT

This paper presents a new regularized boundary integral equation (BIE) method for three-dimensional (3D) potential gradient field. For this method, we firstly construct two special tangential vectors, and then provide a characteristics theorem with respect to the contour integrations of normal and tangential gradients of the fundamental solution. Finally, a new regularized boundary integral equation with indirect unknowns is derived by using the characteristics theorem and a limit theorem. Compared with the direct boundary element method (BEM), the proposed method has three new features: (1) the continuity requirements of density functions are reduced from $C^{1,\alpha}$ to $C^{0,\alpha}$; (2) the BIE does not involve the hypersingular (HFP) integral and thus its numerical evaluation is more easy and precise; (3) any potential gradients on the boundary, not limited to normal gradients, can now be calculated. Numerical results illustrate that the present method is computationally efficient, accurate, and convergent with an increasing number of boundary elements.

1. Introduction

The boundary element method (BEM) has been developed to be a powerful numerical method for solving the potential problems [1–4]. The method reduces the dimensionality of numerical problems by one via an integral transformation of differential equations. However, there exist singularities in the boundary integral equations (BIEs) due to the singular fundamental solution. Therefore, the key of the BEM is how to efficiently deal with the singular integrals [5,6].

Many research efforts have so far been devoted to the accurate and efficient evaluation of singular integrals in the BIEs, and numerous numerical techniques were developed. These methods can be generally divided into two categories: the local strategies [7–20], and the global strategies [21–30]. The former ones include, but are not limited to, analytical and semi-analytical techniques [7,8], new Gaussian quadrature method [9,10], the local regularization method [11–15], transformation method [16,17], finite-part integral [18,19], and subtraction technique [20], etc. Among these methods, the local regularization technique proposed by Guiggiani et al. [11,12] can be used to handle various orders of singularities. For this technique, every quantity in singular integrals has to be extended as Taylor's series about the local distance, which is complex mathematically and not easy to program. Gao [14,15] presented another technique to regularize the singular integrals, in which the singular boundary element is broken up into a few sub-elements. The sub-elements involving the singular point are evaluated analytically to remove the singularities by expressing the non-singular parts of the integration kernels as polynomials of the distance r. The latter ones indirectly calculate the singular integrals by developing new regularized BIEs, such as the virtual BEM [21,22], the null field method [23,24], the simple solution method [25,26], and the regularization methods developed by Zhang et al. [27–30].

In the past, most of researchers focused on regularized BIEs with direct variables. Unlike the direct methods, Zhang [27] was the first who derived an indirect regularized BIE that was not found in previous BEM literature according to the author best knowledge. Recently, the developed regularized method [27] was extended to solving the two-dimensional (2D) anisotropic potential [28] and orthotropic elastic problems [29]. Compared with the direct BEM, the indirect regularized BIEs have the following advantages. First, the continuity requirement of density functions is reduced. In order to remove the singularities, the regularized form

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of the hypersingular (HFP) integral in the direct BEM can be expressed as

$$\int_\Gamma (u(\mathbf{x}) - u(\mathbf{y}) - u_k(\mathbf{x})(x_k - y_k))O\left(\frac{1}{r^2}\right)d\Gamma_{\mathbf{x}},\ k = 1, 2.$$

The existence of this integral in the Riemann sense requires that the density function $u(\mathbf{x})$ belongs to $C^{1,\alpha}$ [31]. However, the existence of the following regularized integral in Ref. [27] only needs that the density function $u(\mathbf{x})$ belongs to $C^{0,\alpha}$

$$\int_\Gamma (u(\mathbf{x}) - u(\mathbf{y}))O\left(\frac{1}{r}\right)d\Gamma_{\mathbf{x}},\ k = 1, 2.$$

Second, the BIE does not involve the hypersingular (HFP) integral and thus its numerical evaluation is more easy and precise. Finally, any potential gradients on the boundary, not limited to the normal gradients, can now be calculated by using the indirect regularized method. In addition, the Galerkin boundary node method (GBNM) [32–34] was developed by combining the moving least-squares approximation and the indirect BIEs. Thus, the new regularized technique for the indirect BIEs will promote the development of the GBNM.

It is not an easy work to extend the approach in Ref. [27] for three-dimensional (3D) problems. For plane problems, there is only a unit tangent vector along the boundary curve, and the contour integrations of tangential gradients about relative quantities are equal to zero. Furthermore, the contour integration of normal gradient can be transformed to that of the tangential gradient by using the relationship $n_1 = t_2, n_2 = -t_1$ of the unit tangent vector $\mathbf{t}(t_1, t_2)$ and gradient vector $\mathbf{n}(n_1, n_2)$. For 3D problems, however, the unit tangent vector is any vector involved in the tangent plane, and the contour integration of any tangential gradient about relative quantities is usually unequal to zero. The contour integration of the normal gradient is also not connected with that of the tangential gradient because of no relationship between the unit tangent vector $\mathbf{t}(t_1, t_2, t_3)$ and gradient vector $\mathbf{n}(n_1, n_2, n_3)$ except $\mathbf{n} \cdot \mathbf{t} = 0$. For these reasons, we were unable to extend the approach in Ref. [27] for 3D problems until now.

In this paper, we present a new indirect regularized BIE for the 3D potential gradient field. Two special tangential vectors are constructed. Then, a characteristics theorem about the contour integrations of normal and tangential gradients of the fundamental solution is provided. Based on the characteristics theorem and a limit theorem, we finally develop a new regularized BIE with indirect unknowns. Numerical results demonstrate the accuracy and efficiency of the proposed method. A brief outline of the paper is as follows. Section 2 presents four theorems. In Section 3, we establish the regularized BIE based on these theorems. In Section 4, the details of boundary elements are given. Section 5 provides numerical examples. In Section 6, we conclude the paper.

2. Basic theorems

In this paper, we always assume that Ω is a bounded domain in R^3, $\Gamma = \partial\Omega$ and $\Omega_c = R^3 - (\Omega \cup \Gamma)$. $\mathbf{n}(\mathbf{x}) = (n_1, n_2, n_3)$ is an outward unit normal vector of Γ at a boundary point $\mathbf{x}$. $\mathbf{m}^1(\mathbf{x}) = (n_2 + kn_3, -n_1, -kn_1)$ and $\mathbf{m}^2(\mathbf{x}) = (n_2, -n_1 + n_3/k, -n_2/k)$ (k is a parameter, and $k \neq 0$) belong to the tangent plane of Γ at the point $\mathbf{x}$. Here, $\mathbf{m}^1(\mathbf{x})$ and $\mathbf{m}^2(\mathbf{x})$ are constructed to compose a linearly independent set with $\mathbf{n}(\mathbf{x})$. Then, the potential gradient in any direction can be expressed as a linear combination of the normal and two tangential gradients. Finally, we can regularize the BIE of potential gradient by using a characteristics theorem about the contour integrations of normal and tangential gradients of the fundamental solution.

The fundamental solution G of 3D potential problem [35] can be expressed as

$$G(\mathbf{x}, \mathbf{y}) = \frac{1}{4\pi r(\mathbf{x}, \mathbf{y})} \tag{1}$$

where $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2, y_3)$ are the field and source points, respectively. $r(\boldsymbol{x}, \boldsymbol{y})$ is the distance between the source and field points. The fundamental solution (1) has an identity as follows:

$$\int_\Gamma \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}(\mathbf{x})} d\Gamma = \begin{cases} -1, & \mathbf{y} \in \Omega \\ 0, & \mathbf{y} \in \Omega_c \end{cases} \tag{2}$$

The proof of this identity can be found in Refs. [36,37]. To derive a new regularized BIE with indirect unknowns, we also present some new identities.

Theorem 1. *Let k satisfies the following two conditions:*

(1) *If $n_1 \neq 0$ and $n_3^2 + 4n_1n_2 \geq 0$, then $k \neq \frac{n_3 \pm \sqrt{n_3^2 + 4n_1n_2}}{2n_1}$;*
(2) *If $n_1 = 0$, $n_3 \neq 0$, then $k \neq -n_2/n_3$.*
Then, $(\mathbf{m}^1, \mathbf{m}^2, \mathbf{n})$ is a linearly independent set.

Proof. Assume $\mathbf{m}^1, \mathbf{m}^2, \mathbf{n}$ are linearly dependent, we will hold

$$D = \begin{vmatrix} n_2 + kn_3 & n_2 & n_1 \\ -n_1 & -n_1 + \frac{1}{k}n_3 & n_2 \\ -kn_1 & -\frac{1}{k}n_2 & n_3 \end{vmatrix} = -kn_1 + \frac{1}{k}n_2 + n_3 = 0 \tag{3}$$

Eq. (3) can be transformed as

$$n_1k^2 - n_3k - n_2 = 0 \tag{4}$$

As $n_1 \neq 0$, the above equation has a solution $k = \frac{n_3 \pm \sqrt{n_3^2 + 4n_1n_2}}{2n_1}$ if and only if $n_3^2 + 4n_1n_2 \geq 0$, and conversely, it has no solution. As $n_1 = 0$, Eq. (4) can be expressed as

$$n_3k + n_2 = 0 \tag{5}$$

If $n_3 \neq 0$, Eq. (5) has solution $k = -n_2/n_3$. As $n_3 = 0$, there is no solution due to $n_1^2 + n_2^2 + n_3^2 = 1$. Therefore, the assumption is false. □

Theorem 2. *Let S be a piecewise smooth surface, and $g(\mathbf{x})$ be a derivable function on S. If $(\mathbf{m}^1, \mathbf{m}^2, \mathbf{n})$ is a linearly independent set, then there holds*

$$\nabla g(\mathbf{x}) = \mathbf{a}(\mathbf{x})\nabla g(\mathbf{x}) \cdot \mathbf{m}^1 + \mathbf{b}(\mathbf{x})\nabla g(\mathbf{x}) \cdot \mathbf{m}^2 + \mathbf{c}(\mathbf{x})\nabla g(\mathbf{x}) \cdot \mathbf{n} \tag{6}$$

where $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$, and $a_i(\mathbf{x})$, $b_i(\mathbf{x})$, $c_i(\mathbf{x})$ $(i = 1, 2, 3)$ are the components of vectors $\mathbf{a}(\mathbf{x}), \mathbf{b}(\mathbf{x}), \mathbf{c}(\mathbf{x})$ respectively, which can be expressed as

$$a_i(\mathbf{x}) = \frac{(\delta_i \times \mathbf{m}^2) \cdot \mathbf{n}}{(\mathbf{m}^1 \times \mathbf{m}^2) \cdot \mathbf{n}}, \quad b_i(\mathbf{x}) = \frac{(\mathbf{m}^1 \times \delta_i) \cdot \mathbf{n}}{(\mathbf{m}^1 \times \mathbf{m}^2) \cdot \mathbf{n}},$$
$$c_i(\mathbf{x}) = \frac{(\mathbf{m}^1 \times \mathbf{m}^2) \cdot \delta_i}{(\mathbf{m}^1 \times \mathbf{m}^2) \cdot \mathbf{n}}, \quad \delta_i = (\delta_{i1}, \delta_{i2}, \delta_{i3}),\ i = 1, 2, 3 \tag{7}$$

Proof. According to Eq. (6), we have

$$\frac{\partial g(\mathbf{x})}{\partial x_i} = a_i(\mathbf{x})\nabla g(\mathbf{x}) \cdot \mathbf{m}^1 + b_i(\mathbf{x})\nabla g(\mathbf{x}) \cdot \mathbf{m}^2 + c_i(\mathbf{x})\nabla g(\mathbf{x}) \cdot \mathbf{n},$$
$$i = 1, 2, 3 \tag{8}$$

and then

$$\delta_{ij} = a_i(\mathbf{x})m_j^1 + b_i(\mathbf{x})m_j^2 + c_i(\mathbf{x})n_j \quad i, j = 1, 2, 3 \tag{9}$$

where $m_j^1, m_j^2, n_j (j = 1, 2, 3)$ are the components of vectors $\mathbf{m}^1, \mathbf{m}^2, \mathbf{n}$. Since $(\mathbf{m}^1, \mathbf{m}^2, \mathbf{n})$ is a linearly independent set, we