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Review of energy conservation errors in finite element softwares caused by using energy-inconsistent objective stress rates

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ABSTRACT

The paper briefly summarizes the theoretical derivation of the objective stress rates that are work-conjugate to various finite strain tensors, and then briefly reviews several practical examples demonstrating large errors that can be used by energy inconsistent stress rates. It is concluded that the software makers should switch to the Truesdell objective stress rate, which is work-conjugate to Green's Lagrangian finite strain tensor. The Jaumann rate of Cauchy stress and the Green-Naghdi rate, currently used in most software, should be abandoned since they are not work-conjugate to any finite strain tensor. The Jaumann rate of Kirchhoff stress is work-conjugate to the Hencky logarithmic strain tensor but, because of an energy inconsistency in the work of initial stresses, can lead to severe errors in the cases of high natural orthotropy or strain-induced incremental orthotropy due to material damage. If the commercial softwares are not revised, the user still can make in the user's implicit or explicit material subroutines (such as UMAT and VUMAT in ABAQUS) a simple transformation of the incremental constitutive relation to the Truesdell rate, and the commercial software then delivers energy consistent results.

1. Introduction

Large deformations of solids are an important practical problem, most challenging for computational predictions [1,2]. The main difficulty is to characterize the rate of stress change at various points of the solid in a way that gives correct work of deformation and describes the material deformation objectively, i.e., independently of the rigid-body rotations material elements.

Commercial softwares such as ABAQUS, LS-DYNA, ANSYS and NASTRAN have traditionally used an objective stress rate or increment which involves a convenient simplification that makes a certain error in energy conservation. For most applications this error is negligible. However, the authors show that large errors, of the order of 30–100%, can arise in certain problems of highly compressible materials, or soft-in-shear highly orthotropic materials, or materials which develop a highly orthotropic damage due to oriented cracking.

This article first explains the concept of energy-consistent objective stress rates. Then it briefly reviews several examples of large errors that can be caused by using commercial codes with an objective stress rate definition that is not energy consistent.

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2. Review of energy-consistent objective stress rates

While the usual way to derive the objective stress rates has been based on tensorial coordinate transformations, the variational energy approach [3] is preferable because it also ensures energy consistency with the finite strain tensor. Consider incremental finite strain tensors ϵ_{ij} relative to the initial (stressed) state at the beginning of the load step, using the initial (Lagrangian) coordinates x_i ($i = 1, 2, 3$) of material points. A broad class of equally admissible finite strain tensors is represented by the Doyle-Ericksen tensors whose second-order approximation is

$$\epsilon_{ij}^{(m)} = e_{ij} + \frac{1}{2} u_{k,i} u_{k,j} - \frac{1}{2}(2 - m) e_{ki} e_{kj} \tag{1}$$

where u_i are the material point displacements, $e_{ij} = (u_{i,j} + u_{j,i})/2$ = small (linearized) strain tensor, and subscripts preceded by a comma denote partial derivatives. The case $m = 2$ gives the Green-Lagrangian strain tensor, $m = 1$ gives the Biot strain tensor, $m = 0$ gives the Hencky (logarithmic) strain tensor, $m = -2$ gives the Almansi-Lagrangian strain tensor. The increments of nonsymmetric small Lagrangian (or first Piola–Kirchhoff) stress τ_{ij} and small stress $\sigma_{ij}^{(m)}$, which is symmetric and objective (an incremental second Piola–Kirchhoff stress), are defined with respect to the Cauchy stress S_{ij}^0 (true stress) in the initial state by the relations

$$T_{ij} = S_{ij}^0 + \tau_{ij} \text{ and } \Sigma_{ij}^{(m)} = S_{ij}^0 + \sigma_{ij}^{(m)} \tag{2}$$

Then the work δW done at small deformations of a material element of unit initial volume can be expressed in two equivalent ways:

$$\delta W = \left(S^0_{ij} + \tau_{ij}\right)\delta u_{i,j} \tag{3}$$

$$\delta W = \left(S^0_{ij} + \sigma^{(m)}_{ij}\right)\delta\epsilon^{(m)}_{ij} \tag{4}$$

where $\delta\epsilon^{(m)}_{ij}$ is the arbitrary variation of incremental finite strain tensor $\epsilon^{(m)}_{ij}$. Since the first-order work $S^0_{ij}\delta u_{i,j}$ is canceled in the virtual work equation of equilibrium by the work of loads, only the second-order work is of interest.

The two work expressions in Eqs. (3) and (4) must be equal. Impose this equality and substitute $S_{ij}\delta u_{i,j} = S_{ij}\delta e_{ij} = S_{ij}\dot{e}_{ij}\Delta t$ (by virtue of symmetry of S_{ij}), $\sigma^{(m)}_{ij}\delta\epsilon^{(m)}_{ij} \approx \sigma^{(m)}_{ij}\dot{e}_{ij}\Delta t$ (which suffices for second-order work accuracy in $u_{i,j}$), $S_{ij}\delta\epsilon^{(m)}_{ij} = S_{pq}(\partial\epsilon^{(m)}_{pq}/\partial u_{i,j})v_{i,j}\Delta t$ and $\sigma^{(m)}_{ij} = \widehat{S}^{(m)}_{ij}\Delta t$ (where $v_{i,j}\ \Delta t = \delta u_{i,j}$, and $v_i = \dot{u}_i$). Then introduce the variational condition that the resulting equation must be valid for any $\delta u_{i,j}$. This yields [3,5]:

$$\widehat{S}^{(m)}_{ij} = \dot{T}_{ij} - S_{pq}\ \frac{\partial^2\left(\epsilon^{(m)}_{pq} - e_{pq}\right)}{\partial t\ \partial u_{i,j}} \tag{5}$$

where $\dot{T}_{ij} = \partial T_{ij}/\partial t = \partial\tau_{ij}/\partial t = \dot{S}_{ij} - S_{ik}v_{j,k} + S_{ij}v_{k,k} = \lim_{\delta t\to 0}\tau_{ij}/\delta t$, $T_{ij} = S^0_{ij} + \tau_{ij}$, and $\dot{S}_{ij} = \partial S_{ij}/\partial t =$ material rate of Cauchy stress. Evaluating Eq. (5) for general m and for $m = 2$, one gets a general expression for the objective stress rate [3,5]:

$$\widehat{S}^{(m)}_{ij} = \widehat{S}^{(2)}_{ij} + \frac{1}{2}(2-m)(S_{ik}\dot{e}_{kj} + S_{jk}\dot{e}_{ki}) \tag{6}$$

where $\widehat{S}^{(2)}_{ij} = \dot{S}_{ij} - S_{kj}v_{i,k} - S_{ki}v_{j,k} + S_{ij}v_{k,k} =$ Truesdell rate. For $m = 2$, Eq. (5) reduces to the Truesdell rate. For $m = 1$ it gives the Biot rate. For $m = 0$, Eq. (5) gives the Jaumann rate of Kirchhoff stress,

$$\widehat{S}^{(0)}_{ij} = \dot{S}_{ij} - \dot{\omega}_{ik}S_{kj} - S_{ik}\dot{\omega}_{kj} + S_{ij}v_{k,k} \tag{7}$$

This rate is work-conjugate to the Hencky (or logarithmic) strain. The Jaumann (or co-rotational) rate of Cauchy stress cannot be obtained from Eq. (5) and thus is work-conjugate with no finite strain tensor.

When different m are considered, the tangential stress–strain relation must be written as $\widehat{S}^{(m)}_{ij} = C^{(m)}_{ijkl}\dot{e}_{kl}$ where moduli $C^{(m)}_{ijkl}$ are associated with strain tensor $\epsilon^{(m)}_{ij}$. They are different for different choices of m, and are related as follows [3,5]:

$$C^{(m)}_{ijkl} = C^{(2)}_{ijkl} + (2-m)[S_{ik}\delta_{jl}]_{sym} \tag{8}$$

$$[S_{ik}\delta_{jl}]_{sym} = \frac{1}{4}(S_{ik}\delta_{jl} + S_{jk}\delta_{il} + S_{il}\delta_{jk} + S_{jl}\delta_{ik}) \tag{9}$$

Here $C^{(2)}_{ijkl}$ are the tangential moduli associated with the Green-Lagrangian strain ($m = 2$), taken as a reference; $S_{ij} =$ current Cauchy stress, and δ_{ij} = Kronecker delta. Using Eq. (9) in each finite element in each loading step, one can convert a black-box commercial finite element program from one objective stress rate to another (this is done in the user's material subroutine of the commercial software).

3. Errors caused by energy inconsistency

Many finite element softwares utilize stress rates that are not associated with any finite strain. Although this has been no problem for the vast majority of applications to metals, enormous errors can result in some cases. Other errors can arise even if an energy consistent stress rate is employed, because of an improper choice of the finite strain measure. As shown in [3], $m = 2$ is required for all the situations where the tangential moduli are highly orthotropic and the dominant compressive principal stress has the direction of strong orthotropy. Thus, e.g., $m = 2$ needs to be used for polymers reinforced by unidirectional or bidirectional stiff fibers (note that, on the other hand, $m = -2$ is required when the maximum compressive stress is normal to the strong orthotropy directions as, e.g., for elastomeric bridge or seismic isolation bearings, and for other principal stress ratios the correct m value lies between -2 and 2; see [4, Eq. (29)]).

This paper reviews several recent studies of this problem and gives three examples of the error caused by the wrong use or definition of the objective stress rate and the associated finite strain tensor.

3.1. Stability of sandwich structures

A salient characteristic of sandwich plates is that the shear strain in a soft core is important for buckling. The shear buckling is a problem with a hundred-year controversial history. It requires using the stability criteria for a three-dimensional continuum, which were for half a century a subject of polemics. Although the polemics were resolved four decades ago, some authors still dispute various aspects. All the historical controversies can be traced to the arbitrariness in choosing the finite strain measure and to inattention to the work-conjugacy requirement. This requirement means that the (doubly contracted) product of the incremental objective stress tensor with the incremental finite strain tensor must give a correct expression for the second-order work [5, chapter 11].

As an example, the cylindrical buckling, which is a special case of plate buckling, is analyzed. The short sides of the sandwich panel are clamped and the longer edges are free; Fig. 1. The core is assumed to be linear elastic and the skins are elastic and quasi-isotropic. The material properties are summarized in Table 1. For numerical simulation, the plate is homogenized through its whole thickness and uniform effective material properties for the combined thickness of the core and skins are used. This defines a homogeneous highly orthotropic plate [6].

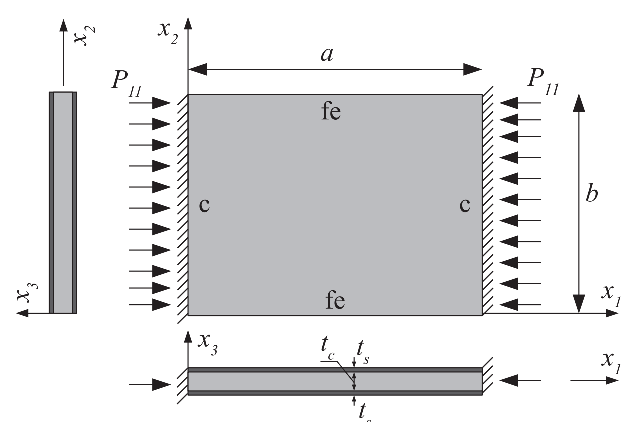

Fig. 1. Plate analyzed: both edges perpendicular to the axis x_1 are clamped (c) and the longer edges are not supported (fe).

Table 1
Material properties (m – measured, c – calculated, l – lower bounds from technical specifications [7]).

		E (GPa)	ν (–)	G (GPa)
CFRP (skins)	In-plane	46^m	0.3^m (ν_{12})	17.7^m
	Transversal	5.7^c	0.24^c (ν_{13})	2.0^c
H200 (core)		0.23^l	0.353^l	0.085^l