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A local extrapolation method for finite elements

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ABSTRACT

In most numerical analyses using the Finite Element Method, several quantities, such as stresses, strains, fluid velocities and gradients, are computed at points in the interior of the solid elements, such as Gauss integration points for instance. Nevertheless, in many applications it is necessary to extrapolate these values to nodal points. That is the case with most visualization tools and post-processors, also in programs with auto-adaptive meshes, large deformations schemes such as Arbitrary Lagrangian–Eulerian Methods, and in programs using the Dynamic Programming Method. A generic methodology to perform this extrapolation in a precise and efficient way is proposed.

1. Introduction

Most boundary value problems in Engineering may be formulated with the Finite Element Method using an approach similar to the displacement formulation [1]. In these finite elements analyses, primary variables, such as displacements in equilibrium problems and water head in seepage problems, are computed at the nodes of the elements. However, secondary or derivate variables are computed at internal points rather than at the nodal points. Such variables may be, for instance, the components of stresses and strain, or the fluid velocities and gradients.

The internal points at which the secondary variables are computed include the center of the element and/or the numerical integration points. The most widely adopted integration points are those given by the Gauss–Legendre quadrature. These points are supposed to give the best sampling position for the secondary variables [2].

In many applications, however, it is necessary to evaluate the values of secondary variables at the nodal points. That is the case with most graphical finite element (FE) post-processors. They use the nodal values to compute the variables in a grid of points inside the element, using the interpolation functions of that element. This grid is later used to draw iso-lines or contour plots. Another important application is related to auto-adaptive meshing. The values computed at the Gauss points of a mesh have to be somehow transported to the integration points of a new mesh. Nodal stress values are necessary to compute some error measure used to define the degree of refinement necessary in a certain region of the domain [3]. Other application which requires nodal stress values is related to the Dynamic Programming Method (DPM). DPM may be used to search for the critical surface in slope stability problems, using the stress field produced (at the Gauss points) in a previous FE analyses. These values have to be extrapolated to a grid of points used in a minimization procedure [4–6].

Other numerical methods such as Arbitrary Lagrangian–Eulerian FE (ALE), the Material Point Method (MPM) and its generalization (GIMP) require computation of derivate variables at nodal points. In ALE, convective terms such as spatial derivatives need to be defined as a continuous field over the domain. For instance, stresses are computed at the integration points, and the stress field is generally discontinuous across element edges. In this case the stresses at the integration points are extrapolated using a least square approximation to obtain the nodal stresses [7]. The MPM uses a background calculation grid where any variable can be expressed as a piecewise continuous function by using standard finite element shape functions. In each time step positions, velocities, stresses and strains are updated to define a new state in a deformed background grid. This background grid is then discarded and a new one is created for the next time step. At the start of the next step, the velocities must be calculated by extrapolation from the material points to the new grid. Since there are always more material points than grid vertices, this extrapolation is achieved using a Least Square Method [8]. This extrapolation procedure in MPM requires inversion of a large mass matrix. For a more efficient algorithm, a local extrapolation could be used.

Mathematically, values at the nodal points could be evaluated directly, i.e., in the same way as those computed at the Gauss points. However, points close the border of a finite element are the worst sampling points and might produce serious errors, since

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equilibrium is satisfied only in an average sense when using weak formulations (displacement approach). The existence of optimal sampling points has been suggested and investigated numerically and mathematically by many authors [9–11]. These points might coincide with the ones given by Gauss–Legendre quadrature in some elements, but this is not always the case. If these optimal points or other points with good overall accuracy are known, then it is intuitive that some extrapolation from these values to the nodal points, plus some smoothing procedure such as averaging, might give more accurate solutions than those obtained directly from the displacement formulation. This idea produced several nodal stress recovery algorithms such as the original work proposed by Hinton and Campbell [12]. More details may be found in [13].

Stress recovery procedures may be applied in a local (element) area or globally, i.e., over the whole finite element mesh [14,15]. Global methods are considerably more costly and most efforts have been concentrated in refining local procedures. Among these, perhaps the most popular is the Superconvergent Patch Recovery (SPR) method introduced by Zienkiewicz and Zhu [16,17]. In this technique, the smoothed stresses corresponding to a fixed nodal point are obtained from a polynomial expansion defined over a "patch of elements" sharing the node. This polynomial is determined from a minimum square fitting using the calculated stresses at the surrounding superconvergent points. Usually, the superconvergent points are the same as the Gauss integration points and the polynomial expansion has the same complete order as the shape functions. Contrary to the original local methods focused in an element rather than a patch of elements, the SPR method requires the extra identification of the elements and the corresponding superconvengent points surrounding the particular nodal point for which the stresses are desired. Moreover, a system of equations must be solved for each nodal point individually. Therefore, in the present paper, the authors propose a method that enhances and generalizes the local recovery process over an element. The proposed method can be eventually modified to work over a patch of elements although this extension is not presented here.

Local recovery procedures, such as proposed by Hinton and Campbell [12], work only for certain element types and particular cases, as for instance, when the number of Gauss points is greater or equal to the number of nodal points. On the other hand, it is not uncommon the use of reduced integration in many engineering applications. In fact, eight node quadrilateral elements (Q8 Serendipity type element), with 2×2 Gaussian integration (four points), are perhaps the most widely used in two-dimensional problems. Also it is common to use brick elements of 20 nodes with $2 \times 2 \times 2$ integration in three-dimensional (3D) finite element applications. If someone wishes to extrapolate the stress components computed at the four Gauss points of a Q8 element to its nodes, an efficient scheme should be used, since the problem is underdetermined. This problem is illustrated in Fig. 1.

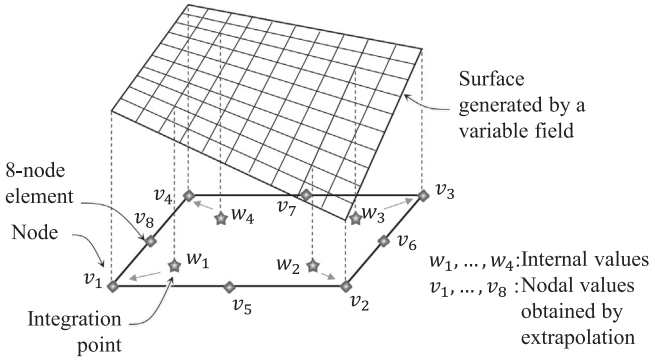

Fig. 1. Extrapolation of internal quantities to nodes in a Q8 element.

Therefore, the main objective of this paper is to present a local extrapolation methodology, which might be generically applied to quantities computed at any number of internal points of finite elements with different shapes. The proposed scheme shows accuracy, efficiency and it can be easily implemented in any computer program.

2. Extrapolation methodology

In some cases, as depicted in Fig. 1, there might be infinite combinations of nodal values that produce, upon interpolation, the desired set of computed values at Gauss integration points. Therefore, the extrapolation of Gauss point values to nodal values may be underdetermined. Yet, it is possible to devise extrapolation schemes, which fit the sampling points and keep a smooth trend over the whole element domain. For the development of such a scheme the following requirements should be fulfilled:

i. Extrapolation should be independent of element type or stress state, i.e., it should be applicable to any type of element either in two or three dimensions (2D or 3D).
ii. The scheme should be valid for any number of nodes and any number of internal sampling points (i.e. number of integration points). The number of nodes could be greater, equal or less than sampling points.
iii. The scheme should minimize the error when the extrapolated nodal point values are interpolated back to the sampling points.
iv. The procedure should keep a smooth variation inside the element and not produce spurious nodal values.

In the following let m denote the number of sampling (Gauss) points and n denote the number of nodal points. By the way, the sampling points may also lie along the sides of the elements. Condition (ii) will be divided in two general cases: (1) $m \geqslant n$; or (2) $m < n$. These cases are treated separately in the next sections.

In the derivation below, it is used the concept of generalized inverse matrix attributed to Moore and Penrose, see e.g. [18,19]. This concept indicates that for a given non-square matrix $\mathbf{A}$, its generalized inverse $\mathbf{A}^+$ can be calculated by the following equation, when the number of lines is greater than the number of columns:

$$\mathbf{A}^+ = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T, \tag{1}$$

and by the following expression when the number of columns is greater than the number of rows:

$$\mathbf{A}^+ = \mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1}. \tag{2}$$

In addition, the sample points values are denoted by w_i and the resultant nodal values are denoted by v_i.

2.1. Case 1: number of sampling points greater than or equal to the number of nodes ($m \geqslant n$)

For the sake of generality, let w denote the value of any quantity evaluated at any point with natural coordinates (ξ, η, ζ) inside a finite element. This value can be approximated by a weighted average of nodal values v_i, using the interpolation functions of the element. This approximation is denoted by $\bar{w}$ and is expressed in the following equation:

$$\bar{w}(\xi,\eta,\zeta) = \sum_{i=1}^{n} N_i(\xi,\eta,\zeta)\, v_i \tag{3}$$