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Computational techniques for solving differential equations by quadratic, quartic and octic spline

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Abstract

The purpose of this paper is to survey the various spline techniques used to solve the differential equations. We have focused on three main techniques namely quadratic, quartic and octic spline techniques used to solve ordinary differential equations of different orders. Here, mostly, we have considered the research papers within last 5 years, so as to get the knowledge of present scenario. Comparisons of methods with own critical comments as remarks have been included.

Keywords: Quadratic spline; Quartic spline; Octic spline; Obstacle; Unilateral; Moving and free boundary-value problems

1. Introduction

In general, it is not possible to obtain the analytical solution of a system of differential equations, obtained from obstacle, unilateral, moving and free boundary-value problems and problems of the defection of plates and in a number of other scientific applications. In the present paper, we discuss about the application of spline functions to solve these boundary-value problems. Our special emphasis is on quadratic and quartic splines.

We include here recent papers based on application of quadratic and quartic spline functions to solve various systems of differential equations. The paper is organized as follows: In Section 2, we will discuss brief definitions of spline, quadratic spline, quartic spline and octic spline. In Section 3, we consider the papers having quadratic spline techniques to solve boundary-value problems. In Section 4, we discuss about quartic spline techniques to solve BVPs. In Section 5, octic spline technique is discussed. In Section 6, the conclusion and further development is given.

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2. Spline functions

Splines exhibit the maximum possible smoothness. Usually a spline is a piecewise polynomial function defined in a region D, such that there exists a decomposition of D into sub-regions in each of which the function is a polynomial of some degree m. Also the function, as a rule, is continuous in D, together with its derivatives of order up to $(m-1)$.

A quadratic spline function $S_\Delta(x)$, interpolating to a function $u(x)$ defined on $[a,b]$ is such that

(i) In each subinterval $[x_{j-1},x_j]$, $S_\Delta(x)$ is a polynomial of degree at most two.
(ii) The first derivative of $S_\Delta(x)$ is continuous on $[a,b]$.

A quartic spline function $S_\Delta(x)$, interpolating to a function $u(x)$ defined on $[a,b]$ is such that

(i) In each subinterval $[x_{j-1},x_j]$, $S_\Delta(x)$ is a polynomial of degree at most four.
(ii) The first, second and third derivatives of $S_\Delta(x)$ are continuous on $[a,b]$.

An octic spline function $S_\Delta(x)$, interpolating to a function $u(x)$ on $[a,b]$ defined as

(i) In each interval $[x_{j-1}, x_j]$, $S_\Delta(x)$ is a polynomial of degree at most eight.
(ii) The first seventh derivatives of $S_\Delta(x)$ are continuous on $[a, b]$.
(iii) $S_\Delta(x_i) = u(x_i)$, $i = 0(1)N+1$.

3. Quadratic spline techniques to solve boundary-value problems

(1) Considering the paper [1] having the system of second-order boundary-value problem of the type

$$y'' = \begin{cases} f(x), & a \leqslant x \leqslant c, \\ g(x)y(x) + f(x) + r, & c \leqslant x \leqslant d, \\ f(x), & d \leqslant x \leqslant b \end{cases} \tag{1.1}$$

with the boundary conditions

$$y(a) = \alpha_1, \quad y(b) = \alpha_2 \tag{1.2}$$

and assuming the continuity conditions of y and y' at c and d. Here, f and g are continuous functions on $[a, b]$ and $[c, d]$, respectively. The parameters r, α_1, α_2 are real finite constants.

In this paper, quadratic non-polynomial spline functions are used to develop a numerical method for obtaining smooth approximations to the solution of a system of second-order boundary-value problems of the type (1.1). The new method is of order two for arbitrary α and β if $2\alpha + 2\beta - 1 = 0$ and method of order 4 if $\alpha = 1/12$ along with $2\alpha + 2\beta - 1 = 0$.

As evident form [2,3] the quadratic and cubic polynomial spline functions need three and four coefficients evaluation at each subinterval and can produce a numerical scheme of second-order accuracy where as quadratic non-polynomial spline functions need three coefficients evaluation at each subinterval $[x_i, x_{i+1}]$ and produces a fourth-order scheme with a less computational cost. This improvement is because of introduction of parameter k in the trigonometric part of T_2 given below. The spline function they proposed in this paper has the form

$$T_2 = \text{span}\{1, \cos kx, \sin kx\},$$

where k is the frequency of the trigonometric part of the splines function which can be real or pure imaginary and which will be used to raise the accuracy of the method. Thus in each subinterval $x_i \leqslant x \leqslant x_{i+1}$, we have

$$\text{span}\{1, \sin x, \cos x\}$$

or

$$\text{span}\{1, \sinh x, \cosh x\}$$

or

$$\text{span}\{1, x, x^2\}, \quad \text{when } k = 0.$$

This fact is evident when correlation between polynomial and non-polynomial spline basis functions is investigated in the following manner:

$$T_6 = \text{span}\{1, \sin kx, \cos kx\},$$
$$= \text{span}\left\{1, \frac{1}{k}\sin(kx), \frac{2}{k^2}(1 - \cos(kx))\right\}. \tag{1.3}$$

The main idea is to use the condition of continuity to get recurrence relation for (1.1). The advantage of new method is higher accuracy with the less computational effort. In comparison with the finite-difference methods, spline solution has its own advantages. For example, once the solution has been computed, the information required for spline interpolation between mesh points is available. This is particularly important when the solution of the boundary-value problem is required at different locations in the interval $[a, b]$. This approach has the added advantage that it not only provides continuous approximations to$y(x)$, but also for y', y'' and higher derivatives at every point of the range of integration. Also, the C^∞-differentiability of the trigonometric part of non-polynomial splines compensates for the loss of smoothness inherited by polynomial splines. The new method performs better than the other collocation, finite-difference and spline methods of same order and thus represents an improvement over existing methods [2–7]. Convergence analysis of the method, numerical validation and comparison with other known methods are also discussed in the paper.

Remark 1. Noor and Khalifa [5] have solved problem (1.1) using collocation method with cubic splines as basis functions. They have shown that this collocation method gives approximation with first-order accuracy. Similar conclusions are pointed out by Noor and Tirmzi [6], where second-order finite-difference methods are used to solve problem (1.1).

Remark 2. Al-Said [2,3] has developed and analyzed quadratic and cubic splines for solving (1.1). He proved that both quadratic and cubic splines methods can be used to produce second-order smooth approximation for the solution of (1.1) and its first derivative over the whole range of integration. More recently, Siraj-ul-Islam et al. [7,8] have established and analyzed optimal smooth approximations for systems second, third-order boundary-value problems and a class of methods for special fourth-boundary-value problems based on cubic, quartic and sextic non-polynomial splines and which provides bases for this approach.

(2) Considering the paper [9] having the system of the linear two point boundary-value problem described by the operator equation of the type:

$$Lu \equiv ru'' + pu' + qu = g \quad \text{in } \Omega \equiv (0, 1), \tag{2.1}$$

where r, p, q and g are given functions of x, and u is the unknown function of x, subject to boundary conditions on the boundary $\partial\Omega$ of Ω, described by

$$Bu \equiv \{\alpha_0 u(0) + \beta_0 u'(0) = \gamma_0, \alpha_1 u(1) + \beta_1 u'(1) = \gamma_1\}, \tag{2.2}$$

where α_i, β_i and γ_i, $i = 0, 1$ are given scalars. The authors focused the discussion to the above BVP (2.1), but some