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On the numerical modeling of convection-diffusion problems by finite element multigrid preconditioning methods

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ABSTRACT

During the last decades, multigrid methods have been extensively used in order to solve large scale linear systems derived from the discretization of partial differential equations using the finite difference method. The effectiveness of the multigrid method can be also exploited by using the finite element method. Finite Element Approximate Inverses in conjunction with Richardson's iterative method could be used as smoothers in the multigrid method. Thus, a new class of smoothers based on approximate inverses can be derived. Effectiveness of explicit approximate inverses relies in the fact that they are close approximants to the inverse of the coefficient matrix and are fast to compute in parallel. Furthermore, the proposed class of finite element approximate inverses in conjunction with the explicit preconditioned Richardson method yield improved results against the classic smoothers such as Jacobi method. Moreover, a dynamic relaxation scheme is proposed based on the Dynamic Over/Under Relaxation (DOUR) algorithm. Furthermore, results for multigrid preconditioned Krylov subspace methods, such as GMRES(res), IDR(s) and BiCGSTAB based on approximate inverse smoothing and a dynamic relaxation technique are presented for the steady-state convection-diffusion equation.

1. Introduction

Let us consider a class of problems defined by the following Partial Differential Equation (P.D.E.):

$$-\varepsilon(\Delta u) + \alpha\frac{\partial u}{\partial x} = f(x,y), \quad (x,y) \in \Omega, \tag{1}$$

where ε is the diffusion coefficient, α is the convection coefficient and f is the force function, cf. [15,18], subject to generalized boundary conditions:

$$\eta(c_1\nabla u) + c_2 u = c_3, \quad (x,y) \in \partial\Omega, \tag{1.a}$$

where Ω is a closed bounded domain, $\partial\Omega$ denotes the boundary of Ω and $\vec{\eta}$ is the outward unit length normal. The c_1, c_2 and c_3 are sufficiently smooth functions in two space variables. Assuming that the associated bilinear form $B(u,v)$ is continuous and coercive on $H^p(\Omega)$ then there exists a solution u_h associated with the solution u of the original problem (1):

$$B(u_h, v_h) = \langle f, v_h\rangle, \quad \forall v_h \in S^h \subset H^p(\Omega), \tag{2}$$

where S^h is a space, p is the order of the Sobolev space and the finite element approximation u_h to u can be derived. The region Ω is then divided into a non-overlapping triangular finite elements of mesh size h with k nodes for each element. Let us consider n nodes in $\Omega \cup \partial\Omega$ not containing points on which the boundary conditions are imposed. The finite element solution over the total elements of the region in a column-wise ordering, is

$$u_h(x,y) = \sum_{i=1}^{n} u_i \Phi_i(x,y), \tag{3}$$

where $\Phi_i(x,y)$ are trial functions. It is well known, that the solution to the bilinear form, viz.,

$$\sum_{i=1}^{n} u_i B(\Phi_i, \Phi_j) = \langle f, \Phi_j\rangle, \quad j = 1,\ldots,n \tag{4}$$

results in the following sparse linear system, i.e.

$$Au = f \tag{5}$$

where the coefficient matrix A is a nonsingular large, sparse, unsymmetric, positive definite matrix of certain structure with semi-bandwidth m, while u is the FE solution at the nodal points and f is a vector, with components resulting from the combination of source terms and imposed boundary conditions, cf. [10,16,17]. The ordering of the grid points is lexicographical. In the case of triangular elements the width of the bands at semi-bandwidth m is two, while in the case of rectangular elements the width of the bands at semi-bandwidth m is three.

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Explicit preconditioned methods have been extensively used for solving sparse linear systems on multiprocessor systems, and the preconditioned form of the sparse linear system (5) is

$$\mathrm{MA}u = \mathrm{M}f \tag{6}$$

where M is a suitable preconditioner. The preconditioner M has to satisfy the following conditions: (i) MA should have a clustered spectrum, (ii) M can be efficiently computed in parallel, (iii) "M × vector" should be fast to compute in parallel. The effectiveness of explicit approximate inverse preconditioning relies on the use of suitable preconditioners that are close approximants to the inverse of the coefficient matrix and are fast to compute in parallel, cf. [9]. In this article we present a parameterized "smoother" based on the explicit approximate inverse matrix and the explicit preconditioned Richardson iterative method.

During the last decade, multigrid methods, have been extensively used, cf. [1,3,5,12,13,18], for solving large sparse linear systems, and gained substantial interest among the scientific community for both their efficiency and convergence behavior.

Multigrid methods are based on the observation that the high frequency components of the error are damped effectively by a stationary iterative method (such as Jacobi or Gauss–Seidel), however the low-frequency components are not damped effectively. In order for low frequency components of the error to be handled, a series of coarser grids with higher mesh size, using triangular and rectangular elements, are used as shown in Fig. 1. In this series of coarser grids the low-frequency modes of the error are more oscillatory and can be damped efficiently by a stationary iterative method, cf. [1,3,13,14,18]. Multigrid methods are composed by four discrete elements: stationary iterative method, restriction operator, prolongation operator and cycle strategy. The stationary iterative methods are first order iterative methods such as Richardson, Jacobi and Gauss–Seidel method. Restriction and prolongation are transfer operators from finer to coarser grids and from coarser to finer grids respectively. The cycle strategy refers to the sequence in which the grids are visited until a solution with the prescribed tolerance is obtained.

2. Multigrid methodology

Let us consider the linear systems derived from the discretization of a PDE at different levels, required by the multigrid method, on a unit square domain with different mesh size h:

$$\mathrm{A}_h u_h = f_h, \tag{7}$$

The linear system (7) can be solved iteratively with a multigrid method. A multigrid method can be formulated by the recursive call of the two-grid method, with v_1 pre-smoothing steps and v_2 post-smoothing steps as well as a correction on the coarser grid, cf. [3,13,18].

An important component in multigrid methodology is a stationary iterative solver, namely "smoother", expressed by the following relation:

$$u_\ell^{(k+1)} = u_\ell^k + \mathrm{M}_\ell r_\ell, r_\ell = f_\ell - \mathrm{A}_\ell u_\ell^{(k)}, \tag{8}$$

where ℓ is the level of discretization and f_ℓ, A_ℓ are the right hand side and the coefficient matrix and $u_\ell^{(k)}$ is the solution vector at the kth iterative step. Further discussions and proofs about classical smoothers can be found in [3,13,14,18]

Approximate inverses in conjunction with the general iterative method (8) can be used as smoothers for multigrid schemes, by considering $\mathrm{M}_\ell = (\mathrm{M}_\ell)_r^{\delta l}$, where $(\mathrm{M}_\ell)_r^{\delta l}$ is a class of finite element approximate inverses of the coefficient matrix of the linear system corresponding to each level of the multigrid method. The finite element approximate inverse matrix $(M_\ell)_r^{\delta l}$, where δl is the "retention" parameter, with $\delta l = \rho m$, $\rho = 1, 2, \ldots, m-1$, can be computed by Optimized Banded Generalized Approximate Inverse Finite Element Matrix (OBGAIFEM) algorithm, cf. [17], based on Finite Element Approximate $L_r U_r$ Factorization (FEALUFA) algorithm, where r is the "fill-in" parameter, i.e. the number of outermost off-diagonal entries retained at semi-bandwidth m in the upper and lower decomposition factors, cf. [16]. The "retention" parameter δl denotes the number of elements retained next to the main diagonal elements of the approximate inverse. The new class of smoothing schemes can be described as follows:

$$u_\ell^{(k+1)} = u_\ell^{(k)} + \omega(\mathrm{M}_\ell)_r^{\delta l}\left(f_\ell - \mathrm{A}_\ell u_\ell^{(k)}\right), \tag{9}$$

where ω is the damping parameter with $0 < \omega \leqslant 1$. In order for a stationary method to function as smoother, the smoothing property must be satisfied, cf. [13,14] and has been proven in [6].

The choice of the relaxation parameter for the approximate inverse smoothing scheme is non-trivial and the DOUR scheme, cf. [12], is used to compute the optimal value of ω. The proposed scheme in conjunction with the DOUR method can be expressed as follows:

$$u_\ell^{(k+1)} = u_\ell^{(k)} + \omega_e(\mathrm{M}_\ell)_r^{\delta l}\left(f_\ell - \mathrm{A}_\ell u_\ell^{(k)}\right), \quad \omega_e = \omega(1+\kappa), \tag{10}$$

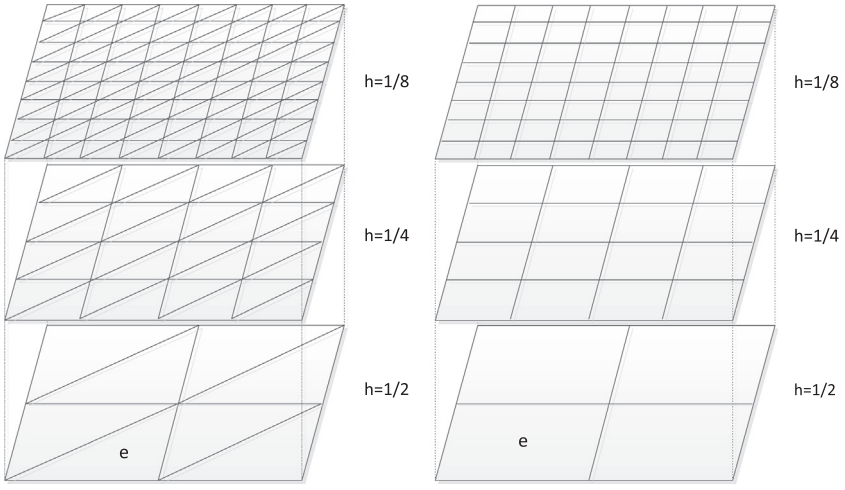

Fig. 1. Square domain discretized with finite element method with different mesh size h using triangular (left) and rectangular (right) elements.