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Boundary point method for linear elasticity using constant and quadratic moving elements

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ABSTRACT

Based on the boundary integral equations and stimulated by the work of Young et al. [J Comput Phys 2005;209:290–321], the boundary point method (BPM) is a newly developed boundary-type meshless method enjoying the favorable features of both the method of fundamental solution (MFS) and the boundary element method (BEM). The present paper extends the BPM to the numerical analysis of linear elasticity. In addition to the constant moving elements, the quadratic moving elements are introduced to improve the accuracy of the stresses near the boundaries in the post processing and to enhance the analysis for thin-wall structures. Numerical tests of the BPM are carried out by benchmark examples in the two- and three-dimensional elasticity. Good agreement is observed between the numerical and the exact solutions.

1. Introduction

For a large class of physical and engineering problems, boundary-type solution methodologies are now well established as viable alternatives to the prevailing domain-type methods such as the finite difference method, the finite element method and finite volume method, because of the computational advantages they offer and its distinctive feature of requiring only the numerical discretisation of the boundary of the solution domain. In particular, boundary-type methods have the merit of predicting accurate and complete solutions while reducing the dimensionality of any given problem by one and thus simplifying the effort involved in data preparation and computer time. The boundary element method (BEM) is the most popular and efficient boundary-type solution procedure, formulated in terms of boundary integral equations (BIE). In the BIE the governing differential equations are converted into integral identities applied over the boundary of the domain. Then the boundary is discretised into small elements in order to carry out the integration.

For elasto-static problems, the variations of displacements and tractions can be described in terms of values at a number of nodal points associated with each element. Shape functions of linear, quadratic or higher order are used for interpolating between the nodes. The integrations over the boundary are usually performed by the Gauss quadrature technique. Details of the BEM can be found in various publications [1,2]. However, the use of elements in the BEM with integrations especially in three-dimensions still puts bourdon on computing efficiency. For example, the BEM requires polygonisation of the boundary surfaces in general 3D cases, and boundary curves in general 2D cases. The regular, weakly singular, strongly singular, and hyper-singular integrals need be dealt with over boundary segments, which is usually a cumbersome and non-trivial task. The presentation of the boundary contour method [3–6] is an effort to improve the efficiency by transferring surface integrals into line integrals via Stokes' theorem. The boundary node method (BNM) represents a coupling between the BIE and the moving least square approximations [7–9]. Using polynomial or radial basis function (RBF) as basis functions, the point interpolation method has been proposed to construct meshfree shape function with Kronecker delta function properties [10,11] with which the boundary conditions can be easily enforced and coupled with the BIE to construct boundary-type meshfree methods [12]. Remarkable progress has been achieved in solving a wide range of static and dynamic problems for solids and structures.

The Trefftz method is another noteworthy boundary-type meshless method featuring conciseness and ease of performance. The crucial structure of the Trefftz method is the use of a set of trial

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functions, singular or non-singular, which *a priori* satisfies the governing differential equation under consideration [13,14]. In the boundary knot method (BKM), a set of non-singular general solutions is employed as trial functions to solve Helmholtz and convection–diffusion equations [15,16]. The merits of using non-singular trial functions such as T-complete functions [17] or general solutions lie in the fact that the collocation and observation points can be coincident and placed on the real boundary of the problem. However, the system matrix thus formed may be ill-conditioned or the condition number of the matrix could become large [17] so that the scale of solvable problems would be limited.

In the method of fundamental solutions (MFS, sometimes called also the F-Trefftz method, charge simulation method, or singularity method), singular functions are taken as the trial functions [18–23]. Just like the BEM, the MFS are best applicable in situations where a fundamental solution to the partial differential equation in question is known. In such cases, the dimensionality of the discretization is reduced. The MFS has certain advantages over the BEM that stems mostly from the fact that the pointisation of the boundary is needed only, which completely avoids any integral evaluations, and makes no principal difference in coding between the 2D and the 3D cases. Because of the singular nature of fundamental solutions, however, the source points must be placed outside the problem domain to avoid the singularity problem, forming a fictitious non-physical boundary. The location of this artificial boundary represents the most serious problem of the MFS and has to be dealt with heuristically, especially for engineering problems with complicated geometry. If the offset distance is too small, the diagonal coefficients of the system matrix will diverge because of the singular nature of fundamental solutions. On the other hand, if the artificial boundary is distant from the real boundary, the system matrices also become ill-conditioned since the condition number of the influence matrix becomes very large. The location of the source and observation points is vital to the accuracy of the solution when implementing the MFS.

A distinct feature in the Trefftz type method including the BKM and the MFS is that each of the coefficients of system matrices is computed on only *one point*, or named integration-free, compared with element-type methods in which the integration must be performed over elements. This feature, which can be termed *one-point* computing, greatly reduces the computing cost. Very recently, by making use of strongly and hyper-singular fundamental solutions of Laplace equation with an indirect formulation, Young et al. proposed a modified MFS for 2D potential problems [24] in which both the source and observation points are located on the real boundary with a singularity removal technique commonly used in the BIE, thus avoiding the inconvenience of using an artificial boundary completely. The only shortcoming of the method seems to be lie in that the equally spaced nodes have to be used along the boundary since the singularity removal technique depends on the divergence-free properties of the kernels [24].

Enlightened by the above-mentioned work, a novel boundary-type meshless method—the boundary point method (BPM) is developed recently for solving the two- and three-dimensional potential problems [25]. The BPM can be looked as something between the MFS and the BEM. In the BPM, the boundary of the problem domain is discretized by boundary nodes, each node having a territory or support where the field variables are defined. By making use of the properties of fundamental solutions, the coefficients of the system matrix in the BPM are computed according to the distances between the two points, the source and observation points. In the cases when the distances are not small, the integrals of kernel functions are evaluated by *one-point* computing, just like that carried out in the MFS, which consist of the most off-diagonal terms of the system matrix. In the cases when the distances are not large, the integrals of kernel functions are evaluated by Gauss quadrature over territories. If the two points coincide, the integrals are treated by the mature techniques available in the BEM [26–28], which constitute the principal diagonal terms of the system matrix. As the adjacent nodes describe the local features of boundary such as position, curvature and direction, the so-called moving elements are introduced in the BPM [25] by organizing the relevant adjacent nodes tentatively, over which the treatment of singularity and Gauss quadrature can be carried out for evaluating the integrals in the latter two cases, i.e., the coincidence or the small distances between the two points.

The current paper extends the BPM to the numerical analysis of linear elasticity. As the field variables are assumed to be constant over each of the territories or supports [25], the accuracy of the field variables in the domain very close to the boundary need to be improved. However, this can be realized by introducing tentatively the quadratic moving element into the BPM in the present work. The basic formulations of the BIE in elasticity are presented in Section 2 as the starting point with the outline of the BPM given briefly in Section 3. The quadratic moving elements are introduced in Section 4. The numerical examples are tested in Section 5, including the comparisons between *one-point* computing and Gauss quadrature and some benchmark examples are presented in the two- and three-dimensional elasticity, showing the feasibility and accuracy of the proposed method.

2. Basic formulations

Considering a linear elastic domain Ω surrounded by the piecewise smooth boundary Γ free of body force, the equilibrium equation is:

$$\sigma_{ij,j} = 0, \quad \text{in } \Omega, \tag{1}$$

where σ_{ij} is the stress tensor. The corresponding boundary conditions are given by:

$$u_i(x) = \bar{u}_i(x), \quad x \in \Gamma_u, \tag{2}$$

$$\tau_i(x) = \sigma_{ij}(x)n_j = \bar{\tau}_i(x) \quad x \in \Gamma_\sigma, \tag{3}$$

where u_i are the displacements, τ_i the tractions, $\bar{u}_i$ the prescribed displacements on the displacement boundary Γ_u, $\bar{\tau}_i$ the prescribed tractions on the traction boundary Γ_σ, and n_i is the outward unit that is normal to the boundary $\Gamma = \Gamma_\sigma \cup \Gamma_u$. From the method of weighted residuals and the constitutive relations of elasticity [1,2], the direct formulations of integral equations can be written as:

$$\gamma(y)\delta_{ij}u_j(y) = \int_\Gamma \tau_j(x)u^*_{ij}(x,y)d\Gamma(x) - \int_\Gamma u_j(x)\tau^*_{ij}(x,y)d\Gamma(x), \tag{4}$$

$$\gamma(y)\delta_{ij}\sigma_{jk}(y) = \int_\Gamma \tau_j(x)u^*_{ikj}(x,y)d\Gamma(x) - \int_\Gamma u_j(x)\tau^*_{ikj}(x,y)d\Gamma(x), \tag{5}$$

where u^*_{ij} and τ^*_{ij} are the Kelvin's displacement and the traction fundamental solutions, u^*_{ikj} and τ^*_{ikj} the derived displacement and traction fundamental solutions, respectively. γ represents the coefficient of the free term of the BIE depending on where the source point y is located. $\gamma(y) = 1$ if $y \in \Omega$, $\gamma(y) = 0$ if $y \in \overline{\Omega \cup \Gamma}$, $\gamma(y) = 0.5$ if $y \in \Gamma$ which is smooth in the neighborhood of the point y. δ_{ij} is the Kronecker symbol. With the Cauchy's relation $\tau_i = \sigma_{ij}n_j$, Eq. (5) can be written as follows:

$$\gamma(y)\delta_{ij}\tau_j(y) = n_k(y)\left\{\int_\Gamma \tau_j(x)u^*_{ikj}(x,y)d\Gamma(x) - \int_\Gamma u_j(x)\tau^*_{ikj}(x,y)d\Gamma(x)\right\}. \tag{6}$$

It is known that, in Eqs. (4)–(6), when $y \in \Gamma$, the integrals with the kernel u^*_{ij} are weakly singular ($O(\log(r^{-1}))$ for 2D or $O(r^{-1})$ for 3D), the integrals with the kernels τ^*_{ij} and u^*_{ikj} are strongly singular