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The verification of the quantities of interest based on energy norm of solutions by node-based smoothed finite element method

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ABSTRACT

Verification of the quantities of interest computed with the finite element method (FEM) requires an upper bound on the strain energy, which is half of the energy norm of displacement solutions. Recently, a modified finite element method with strain smoothing, the node-based smoothed finite element method (NS-FEM), has been proposed to solve solid mechanics problems. It has been found in some cases that the energy norm formed by the smoothed strain of NS-FEM solutions bounds the energy norm of exact displacements from above. We analyze the bounding property of this method, give three kind of energy norms of solutions computed by FEM and NS-FEM, and extend them to the computation of an upper bound and a lower bound on the linear functional of displacements. By examining the bounding property of NS-FEM with different energy norms using some linear elastic problems, the advantages of NS-FEM over the traditional error estimate based methods is observed.

1. Introduction

One problem with the FEM is the verification of the finite element solutions; this leads to the development of error bound methods for providing the evaluations of the global quality of finite element analyses [1–6]. In engineering design, verification of the computation of the quantities of interest plays an important role in improving the structural design for safety, such as the displacements at a point, the stresses in a local area and the stress intensity factors at crack tips. Some verification methods have been proposed for computing upper and lower bounds on quantities of interest that are the functions of displacements [7–10], and a key ingredient of these methods is the computation of global upper and lower bounds on the total strain energy.

It is known that finite element approximations based on the potential energy principle and the complementary energy principle produce a lower bound and an upper bound on the total strain energy, respectively [11]. The former is referred to as the displacement method, the latter the equilibrium method, which is not commonly used in the commercial softwares for the difficulties arising in handling the boundary conditions [11–14]. In recent years, the methods for the computation of guaranteed upper bound on the energy norm of the exact error in the finite element solutions have been developed [15–20], which provided an alternative approach for finding an upper bound on the strain energy by solving independent subproblems based on elements or patches. In these methods, boundary loads on element edges are needed to construct elemental independent subproblems for solving the equilibrium stresses, or the boundary loads on element edges are waived in constructing the patch based independent subproblems to solve the equilibrium stresses, that is the so called flux-free methods.

Recently, a modified finite element method with strain smoothing, the node-based smoothed finite element method (NS-FEM), has been proposed to solve solid mechanics problems. It has been found that the strain energy computed by NS-FEM bounds the exact strain energy from above in the computational experiments [21–26]. This method was first proposed to develop a stabilized nodal integration scheme for the Galerkin mesh-free method by introducing a strain smoothing stabilization to compute nodal strains by a divergence counterpart of a strain spatial averaging [27].

In this study we analyze the bounding properties of the NS-FEM, and give some inequalities for comparing different energy norms of displacements computed by the FEM and the NS-FEM. We also extend this method to compute the upper and lower bounds on the functions of displacements, and all of the examples are specifically chosen, such as tension, bending problems with different boundary conditions, and the problem with stress concentration, to show the performance of the method.

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This paper is organized as follows, in Section 2 the problem statement and definitions are introduced. In Section 3 the NS-FEM and some properties are briefly described thereof. In Section 4 the upper and lower bounds on the linear functionals of solutions are derived, and some lemmas and remarks to these bounds are given. In Section 5 some numerical experiments are reported and commented on. Finally some conclusions are drawn in the last section.

2. Problem statement and definitions

For simplifying the expression of the method in this paper, we only consider the equations of two dimensional linear elasticity. Let us consider a bounded elastic body $\Omega \subset \mathbb{R}^2$, the boundary of Ω is assumed piecewise smooth, and composed of Dirichlet portion Γ_D and Neumann portion Γ_N, i.e. $\partial\Omega = \overline{\Gamma_D \cup \Gamma_N}$. The strong form of the problem reads: find $\boldsymbol{u}$ in $\mathscr{V} = \{\boldsymbol{v} \in (\mathscr{H}^1(\Omega))^2 | \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_D\}$, such that

$$-\mathscr{D}\boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{f} \text{ in } \Omega,$$

subject to $\boldsymbol{N}\boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{t}$ on Γ_N, where $\mathscr{H}^1(\Omega)$ denotes the usual Sobolev space, $\boldsymbol{u}_D$ the displacement on Γ_D, and

$$\mathscr{D} = \begin{bmatrix} \frac{\partial}{\partial x_1} & 0 & \frac{\partial}{\partial x_2} \\ 0 & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} \end{bmatrix},$$

is the differential operator matrix, $\boldsymbol{f}$ the load, $\boldsymbol{t}$ the prescribed boundary traction. Stress $\boldsymbol{\sigma}(\boldsymbol{u}) = \{\sigma_{11}, \sigma_{22}, \tau_{12}\}^T$ is related to strain $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \mathscr{D}^T\boldsymbol{u} = \{\varepsilon_{11}, \varepsilon_{22}, 2\gamma_{12}\}^T$ by the material law, $\boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{E}\boldsymbol{\varepsilon}(\boldsymbol{u})$, in which $\boldsymbol{E}$ is the matrix of elastic moduli,

$$\boldsymbol{E} = \begin{bmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix},$$

where λ and μ are Lame's constants, $\mu = \frac{E}{2(1+\nu)}$, and $\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$ for plane strain problem, $\lambda = \frac{E\nu}{1-\nu^2}$ for plane stress problem, here E and ν are Young's elastic modulus and Poisson's ratio, respectively. $\boldsymbol{N}$ is the matrix constructed with n_1 and n_2 that are the components of the unit external normal vector $\boldsymbol{n}$ on Γ_N,

$$\boldsymbol{N} = \begin{bmatrix} n_1 & 0 & n_2 \\ 0 & n_2 & n_1 \end{bmatrix}.$$

The weak form of the above equation reads: find $\boldsymbol{u}$ in $\mathscr{V}$ such that

$$\int_\Omega \boldsymbol{\varepsilon}^T(\boldsymbol{u})\boldsymbol{E}\boldsymbol{\varepsilon}(\boldsymbol{v})\, d\Omega = \ell(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \mathscr{V},$$

in which $\ell(\boldsymbol{v}) = \int_\Omega \boldsymbol{f}^{\,T}\boldsymbol{v}\, d\Omega + \int_{\Gamma_N} \boldsymbol{t}^T\boldsymbol{v}\, d\Gamma$. The energy norm associated with the bilinear form $\int_\Omega \boldsymbol{\varepsilon}^T(\cdot)\boldsymbol{E}\boldsymbol{\varepsilon}(\cdot)\, d\Omega$ is defined as

$$\|\boldsymbol{v}\|^2 = \int_\Omega \boldsymbol{\varepsilon}^T(\boldsymbol{v})\boldsymbol{E}\boldsymbol{\varepsilon}(\boldsymbol{v})\, d\Omega.$$

In order to obtain an approximate solution of the weak problem (2), a finite-dimensional counterpart of all these variational forms given above can be built using the Galerkin FEM. We denote $\mathscr{V}_h \subset \mathscr{V}$ the finite element spaces of continuous functions that are piecewise polynomials of degree $r \geqslant 1$. The corresponding finite element solution in $\mathscr{V}_h$ is denoted by $\boldsymbol{u}_h$ and satisfies the equation:

$$\int_\Omega \boldsymbol{\varepsilon}^T(\boldsymbol{u}_h)\boldsymbol{E}\boldsymbol{\varepsilon}(\boldsymbol{v})\, d\Omega = \ell(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \mathscr{V}_h. \tag{1}$$

Now let us consider the output, which is a linear functional of the solution $\boldsymbol{u}$ defined as $\ell^{\mathscr{O}}(\boldsymbol{u})$, i.e. $\ell^{\mathscr{O}} : (\mathscr{H}^1(\Omega))^2 \mapsto \mathbb{R}$. Similar to $\ell(\boldsymbol{u})$, $\ell^{\mathscr{O}}(\boldsymbol{u})$ should has the form as $\ell^{\mathscr{O}}(\boldsymbol{v}) = \int_{\Gamma_{\mathscr{O}}} \boldsymbol{p}^T\boldsymbol{v}\, d\Gamma$. Since the output is used in the right hand side of the dual problem defined as follows, it is required that the outputs depend explicitly on the solution $\boldsymbol{u}$. In order to derive upper and lower bounds on the output $\ell^{\mathscr{O}}(\boldsymbol{u})$, we introduce the following adjoint or dual problem: find $\boldsymbol{u}^D \in \mathscr{V}$ such that

$$\int_\Omega \boldsymbol{\varepsilon}^T(\boldsymbol{v})\boldsymbol{E}\boldsymbol{\varepsilon}(\boldsymbol{u}^D)\, d\Omega = \ell^{\mathscr{O}}(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \mathscr{V}, \tag{2}$$

and the corresponding finite element approximation, $\boldsymbol{u}_h^D \in \mathscr{V}_h \subset \mathscr{V}$, such that

$$\int_\Omega \boldsymbol{\varepsilon}^T(\boldsymbol{v})\boldsymbol{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_h^D)\, d\Omega = \ell^{\mathscr{O}}(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \mathscr{V}_h. \tag{3}$$

In this paper, we use the simplest two dimensional elements, the linear triangular elements, to implement the above primal and dual problems.

3. The NS-FEM

Let us partition the computational domain Ω into smoothing subdomains $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2 \cup \ldots \cup \bar{\Omega}_N$ with $\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j$, where N is the number of finite element nodes (including the nodes on Γ_D) located in the entire computational domain, and for every node $k = 1, \ldots, N$, the smoothing domain Ω_k is obtained by connecting sequentially the mid-edge point to the centroid of the surrounding triangles of the node as shown in Fig. 1.

Given any strain field $\boldsymbol{\varepsilon}$, the smoothed strain field $\hat{\boldsymbol{\varepsilon}}$ on each smoothing domain Ω_k is obtained by a nodal smoothing operation as

$$\hat{\boldsymbol{\varepsilon}}_k = \int_{\Omega_k} \boldsymbol{\omega}_k(\boldsymbol{x} - \boldsymbol{x}_k)\; \boldsymbol{\varepsilon}\, d\Omega,$$

where $\boldsymbol{\omega}_k(\boldsymbol{x})$ is a diagonal matrix of the smoothing function $\omega_k(\boldsymbol{x})$ that is positive and normalized to unity:

$$\int_{\Omega_k} \omega_k(\boldsymbol{x})\, d\Omega \equiv 1.$$

The smoothed strain $\hat{\boldsymbol{\varepsilon}}_k$ is a constant over the smoothing domain Ω_k. For two-dimensional elasticity problems the diagonal matrix can be chosen to be $\boldsymbol{\omega}_k(\boldsymbol{x}) = \text{diag}\{\omega_k(\boldsymbol{x}),\ \omega_k(\boldsymbol{x}),\ \omega_k(\boldsymbol{x})\}$. For simplicity the smoothing function $\omega_k(\boldsymbol{x})$ is taken as

$$\omega_k(\boldsymbol{x} - \boldsymbol{x}_k) = \begin{cases} 1/A_k, & \text{if } \boldsymbol{x} \in \Omega_k, \\ 0, & \text{otherwise}, \end{cases}$$

where $A_k = \int_{\Omega_k} d\Omega$ is the area of the smoothing domain Ω_k. Therefore, the smoothed strain in the smoothing domain Ω_k will be

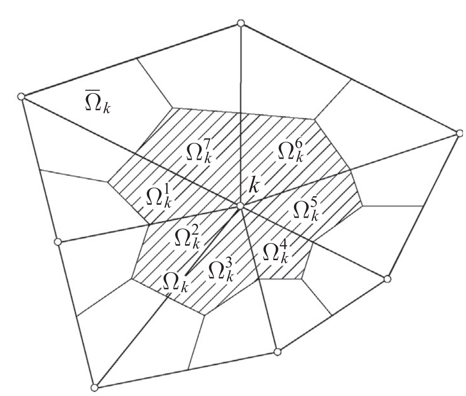

Fig. 1. The finite element mesh and the smoothing domain Ω_k.