Advances in Engineering Software 41 (2010) 13–21

Contents lists available at ScienceDirect

Advances in Engineering Software

journal homepage: www.elsevier.com/locate/advengsoft

Alternating two-stage methods for consistent linear systems with applications to the parallel solution of Markov chains

Héctor Migallón [a], Violeta Migallón [b,*], Jose Penadés [b]

[a] Departamento de Física y Arquitectura de Computadores, Universidad Miguel Hernández, E-03202 Elche, Alicante, Spain
[b] Departamento de Ciencia de la Computación e Inteligencia Artificial, Universidad de Alicante, E-03071 Alicante, Spain

ARTICLE INFO

Article history:
Received 9 October 2007
Accepted 12 January 2008
Available online 25 February 2009

Keywords:
Alternating two-stage methods
Parallel algorithms
Linear systems
Block methods
Markov chains

ABSTRACT

Two-stage methods in which the inner iterations are accomplished by an alternating method are developed. Convergence of these methods is shown in the context of solving singular and nonsingular linear systems. These methods are suitable for parallel computation. Experiments related to finding stationary probability distribution of Markov chains are performed. These experiments demonstrate that the parallel implementation of these methods can solve singular systems of linear equations in substantially less time than the sequential counterparts.

1. Introduction

Consider the $n \times n$ linear system

$$Ax = b, \tag{1}$$

where A is a matrix such that b is in $\mathscr{R}(A)$, the range of A.

Given a splitting $A = M - N$ (M nonsingular), a classical iterative method produces the following iteration

$$x^{(l+1)} = M^{-1}Nx^{(l)} + M^{-1}b, \quad l = 0, 1, \ldots, \tag{2}$$

where $M^{-1}N$ is called the iteration matrix of the method. On the other hand, a two-stage method consists of approximating the linear system (2) by using another iterative procedure (inner iterations). That is, consider the splitting $M = F - G$ and perform, at each outer step $l, s(l)$ inner iterations of the iterative procedure induced by this splitting. Thus, the resulting method is

$$x^{(l+1)} = (F^{-1}G)^{s(l)}x^{(l)} + \sum_{j=0}^{s(l)-1}(F^{-1}G)^j F^{-1}(Nx^{(l)} + b), \quad l = 0, 1, \ldots, \tag{3}$$

cf. [1]. Two-stage iterative methods have been studied, e.g., in [2–5]. In this paper, a two-stage iterative process is developed for the solution of the linear system (1), where at each outer iteration $l, l = 0, 1, \ldots,$ the linear system (2) is approximated by using an alternating iterative procedure. More specifically, let $M = P - Q = R - S$ be two splittings of the matrix M. In order to approximate the linear system (2), for each $l, l = 0, 1, \ldots,$ we perform $s(l)$ inner iterations of the general class of iterative methods of the form

$$z^{(k+\frac{1}{2})} = P^{-1}Qz^{(k)} + P^{-1}(Nx^{(l)} + b),$$
$$z^{(k+1)} = R^{-1}Sz^{(k+\frac{1}{2})} + R^{-1}(Nx^{(l)} + b), \quad k = 0, 1, \ldots, s(l) - 1$$

with $z^{(0)} = x^{(l)}$, or equivalently

$$z^{(k+1)} = R^{-1}SP^{-1}Qz^{(k)} + R^{-1}(SP^{-1} + I)(Nx^{(l)} + b),$$
$$k = 0, 1, \ldots, s(l) - 1.$$

Thus, the alternating two-stage method can be written as follows

$$x^{(l+1)} = (R^{-1}SP^{-1}Q)^{s(l)}x^{(l)} + \sum_{j=0}^{s(l)-1}(R^{-1}SP^{-1}Q)^j R^{-1}(SP^{-1} + I)(Nx^{(l)} + b),$$
$$l = 0, 1, \ldots. \tag{4}$$

In a similar manner as the two-stage methods, we say that an alternating two-stage method is stationary when $s(l) = s$, for all l, while an alternating two-stage method is non-stationary if the number of inner iterations changes with the outer iteration l.

Clearly, given an initial vector $x^{(0)}$, the alternating two-stage iterative method (4) produces the sequence of vectors

$$x^{(l+1)} = T^{(l)}x^{(l)} + c_{s(l)}, \quad l = 0, 1, \ldots, \tag{5}$$

where

$$T^{(l)} = (R^{-1}SP^{-1}Q)^{s(l)} + \sum_{j=0}^{s(l)-1}(R^{-1}SP^{-1}Q)^j R^{-1}(SP^{-1} + I)N, \tag{6}$$

and $c_{s(l)} = \sum_{j=0}^{s(l)-1}(R^{-1}SP^{-1}Q)^j R^{-1}(SP^{-1} + I)b$.

* Corresponding author. Tel.: +34 965 903900; fax: +34 965 903902.
E-mail address: violeta@dccia.ua.es (V. Migallón).

0965-9978/$ - see front matter
doi:10.1016/j.advengsoft.2008.12.021

In order to analyze the convergence of the alternating two-stage method (5) and taking into account that $A = M - N$ and $M = P - Q = R - S$, the iteration matrices $T^{(l)}, l = 0, 1, \ldots,$ defined in (6), are written as follows:

$$\begin{aligned} T^{(l)} &= (R^{-1}SP^{-1}Q)^{s(l)} + \sum_{j=0}^{s(l)-1}(R^{-1}SP^{-1}Q)^j R^{-1}(SP^{-1}+I)N \\ &= (R^{-1}SP^{-1}Q)^{s(l)} + \sum_{j=0}^{s(l)-1}(R^{-1}SP^{-1}Q)^j R^{-1}(SP^{-1}+I)(P-Q)M^{-1}N \\ &= (R^{-1}SP^{-1}Q)^{s(l)} + \sum_{j=0}^{s(l)-1}(R^{-1}SP^{-1}Q)^j R^{-1}S(I-P^{-1}Q)M^{-1}N \\ &\quad + \sum_{j=0}^{s(l)-1}(R^{-1}SP^{-1}Q)^j(I-R^{-1}S)M^{-1}N = (R^{-1}SP^{-1}Q)^{s(l)} \\ &\quad + (I-(R^{-1}SP^{-1}Q)^{s(l)})M^{-1}N, \quad l = 0, 1, \ldots \end{aligned} \tag{7}$$

In this paper, our study concentrates on these alternating two-stage methods. Specifically, in Section 3, we give convergence results of these methods for nonsingular linear systems, when the matrix A of the linear system is monotone, H-matrix or Hermitian positive definite. In Section 4, we also prove the convergence of these methods for consistent singular linear systems, when M-matrices or symmetric positive semidefinite matrices are considered. In Section 5, we explore the use of parallel implementation of these alternating two-stage methods for the solution of Markov chains. Previously, in Section 2, we present some definitions and preliminaries that are used later in the paper.

2. Notation and preliminaries

The notation and terminology adopted in this paper are along the lines of those used by Berman and Plemmons [6]. We say that a vector x is nonnegative (positive), denoted $x \geqslant 0 (x > 0)$, if all of its entries are nonnegative (positive). Similarly, a matrix B is said to be nonnegative, denoted $B \geqslant O$ (where O is the zero matrix), if all its entries are nonnegative. Given a matrix $A = (a_{ij})$, we define the matrix $|A| = (|a_{ij}|)$. It follows that $|A| \geqslant O$ and that $|AB| \leqslant |A||B|$ for any two matrices A and B of compatible size. By $\rho(A)$ we denote the spectral radius of the square matrix A. A general matrix A is called an M-matrix if A can be expressed as $A = sI - B$, with $B \geqslant O$, $s > 0$, and $\rho(B) \leqslant s$. The M-matrix A is singular when $s = \rho(B)$. The M-matrix A is nonsingular when $s > \rho(B)$. Let $Z^{n\times n}$ denote the set of all real $n \times n$ matrices which have all non-positive off-diagonal entries.

A nonsingular matrix $A \in Z^{n\times n}$ is an M-matrix if and only if A is a monotone matrix ($A^{-1} \geqslant O$). For any matrix $A = (a_{ij}) \in \Re^{n\times n}$, we define its comparison matrix $\langle A\rangle = (\alpha_{ij})$ by $\alpha_{ii} = |a_{ii}|, \alpha_{ij} = -|a_{ij}|, i \neq j$. A nonsingular matrix A is said to be an H-matrix if $\langle A\rangle$ is an M-matrix.

Lemma 1 [7,8]. *Let $A, B \in \Re^{n\times n}$.*

(a) *If A is an H-matrix, then $|A^{-1}| \leqslant \langle A\rangle^{-1}$.*
(b) *If $|A| \leqslant B$ then $\rho(A) \leqslant \rho(B)$.*

Definition 2 [6,2,9]). Let $A \in \Re^{n\times n}$. A splitting $A = M - N$ is called

(a) regular if $M^{-1} \geqslant O$ and $N \geqslant O$,
(b) weak regular if $M^{-1} \geqslant O$ and $M^{-1}N \geqslant O$,
(c) *H-splitting* if $\langle M\rangle - |N|$ is a nonsingular M-matrix, and
(d) *H-compatible* splitting if $\langle A\rangle = \langle M\rangle - |N|$.

Lemma 3 [3]. *Given a nonsingular matrix A and a matrix T such that $(I-T)^{-1}$ exists, there is a unique pair of matrices P, Q such that P is nonsingular, $T = P^{-1}Q$ and $A = P - Q$. The matrices are $P = A(I-T)^{-1}$ and $Q = P - A$.*

In the context of Lemma 3, it is said that the unique splitting $A = P - Q$ is induced by the iteration matrix T. We point out that when the matrix A is singular, the induced splitting is not unique; see e.g., [10].

Lemma 4 [6,2]. *Let $A = M - N$ be a splitting.*

(a) *If the splitting is weak regular, then $\rho(M^{-1}N) < 1$ if and only if $A^{-1} \geqslant O$.*
(b) *If the splitting is an H-splitting, then A and M are H-matrices and $\rho(M^{-1}N) \leqslant \rho(\langle M\rangle^{-1}|N|) < 1$.*
(c) *If the splitting is an H-compatible splitting and A is an H-matrix, then it is an H-splitting and thus convergent.*

The transpose and the conjugate transpose of a matrix $A \in \mathbb{C}^{n\times n}$ are denoted by A^T and A^H, respectively. Similarly, given a vector $x \in \mathbb{C}^n$, x^T and x^H denote the transpose and the conjugate transpose of x, respectively. A matrix $A \in \mathbb{C}^{n\times n}$ is said to be symmetric if $A = A^T$, and Hermitian if $A = A^H$. Clearly a real symmetric matrix is a particular case of a Hermitian matrix. A complex, not necessarily Hermitian matrix A, is called positive definite (positive semidefinite) if the real part of x^HAx is positive (nonnegative), for all complex $x \neq 0$. When A is Hermitian, this is equivalent to requiring that $x^HAx > 0 (x^HAx \geqslant 0)$, for all complex $x \neq 0$. A general matrix A is positive definite (positive semidefinite) if and only if the Hermitian matrix $A + A^H$ is positive definite (positive semidefinite). Given a matrix $A \in \mathbb{C}^{n\times n}$, the splitting $A = M - N$ is called P-regular if the matrix $M^H + N$ is positive definite.

Let $T \in \Re^{n\times n}$, by $\sigma(T)$ we denote the spectrum of the matrix T. We define $\gamma(T) = \max\{|\lambda| : \lambda \in \sigma(T), \lambda \neq 1\}$. We say that two subspaces S_1 and S_2 on $\Re^n$ are *complementary* if $S_1 \oplus S_2 = \Re^n$, i.e., if $S_1 \cap S_2 = \{0\}$ and $S_1 + S_2 = \Re^n$. The *index* of a square matrix T, denoted *ind*T, is the smallest nonnegative integer k such that $\mathscr{R}(T^{k+1}) = \mathscr{R}(T^k)$. By ind_1T we denote the *index* associated with the value one, i.e., $ind_1T = ind(I-T)$. Note that when $\rho(T) = 1$, $ind_1T \leqslant 1$ if and only if $ind_1T = 1$. We say that a matrix $T \in \Re^{n\times n}$, is *convergent* if $\lim_{k\to\infty}T^k = O$. It is well known that a matrix T is convergent if and only if $\rho(T) < 1$. By $\mathscr{N}(T)$ we denote the null space of T.

We say that T is *semiconvergent* if $\lim_{k\to\infty}T^k$ exists, although it need not be the zero matrix. If, on the other hand, $\rho(T) = 1$, two different conditions need to be satisfied to guarantee semiconvergence, as the following result shows.

Theorem 5 [11]. *Let $T \in \Re^{n\times n}$, with $\rho(T) = 1$. The matrix T is semiconvergent if and only if the following two statements hold.*
(a) $1 \in \sigma(T)$ and $\gamma(T) < 1$, (b) $\mathscr{N}(I-T) \oplus \mathscr{R}(I-T) = \Re^n$.

Condition (b) is equivalent to the existence of the group inverse $(I-T)^{\#}$, and it is also equivalent to having $ind_1T = 1$; see, e.g., [6]. We review in what follows the definition of some generalized inverses.

Definition 6 [6]. *Let $A \in \Re^{n\times n}$, and consider the following matrix equations.*

(1) $AXA = A$,
(2) $XAX = X$, and
(3) $AX = XA$.

A {1,2}-inverse of A is a matrix X which satisfies conditions (1) and (2). If, in addition, X satisfies condition (3), X is said to be a group inverse of A.

We would like to note that the group inverse $A^{\#}$ of a matrix A, if it exists, is unique. When A is nonsingular, each generalized inverse coincides with A^{-1}.