Advances in Engineering Software 41 (2010) 64–69

Contents lists available at ScienceDirect

Advances in Engineering Software

journal homepage: www.elsevier.com/locate/advengsoft

A variable step size numerical method based on fractional type quadratures for linear integro-differential equations ☆

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ARTICLE INFO

Article history:
Received 14 September 2007
Accepted 2 December 2008
Available online 9 January 2009

Keywords:
Fractional quadratures
Fractional integrals
Variable step

ABSTRACT

In this work a discretization in time of variable step size of the convolution equation $u(t) = \int_0^t (t-s)^{\alpha-1} u(s) \mathrm{d}s$, based on a fractional type quadrature is studied. The convergence is directly proved thanks to a suitable representation of the error by means of Peano kernels. Practical illustrations showing the efficiency of our numerical scheme are provided.

1. Introduction

Integro-differential equations of fractional order have recently showed highly interesting in connection with many fields of science (see [5,9–11]). In fact, we consider linear convolution equations of type

$$u'(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} \lambda u(s) \mathrm{d}s + f(t), \quad u(0) = u_0, \tag{1}$$

where $\lambda \in \mathbb{C}$ and $1 < \alpha < 2$.

It is very well known that the initial value problem (1) can be written in an equivalent way as the integral equation of convolution type

$$u(t) = u_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \lambda u(s) \mathrm{d}s + \int_0^t f(s) \mathrm{d}s, \quad 0 \leqslant t \leqslant T. \tag{2}$$

For the sake of the simplicity, Eq. (2) will be written as

$$u(t) = u_0 + \partial^{-\alpha} \lambda u(t) + \partial^{-1} f(t), \quad 0 \leqslant t \leqslant T, \tag{3}$$

where $\partial^{-\beta}$, for $\beta > 0$, stands for the Riemann–Liouville operator, i.e.,

$$\partial^{-\beta} g(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g(s) \mathrm{d}s, \quad t \geqslant 0,$$

which is understood as the fractional integral of order $\beta > 0$ of the function g. A large variety of properties of fractional integrals and derivatives can be found in [5,10].

Our interest focuses on the numerical solution of (3). In fact, it is easily understandable that the obtention of a numerical scheme to approximate the solution of (3) reduces to a suitable choice of the quadrature to approximate the integral term as we show below.

Let us recall that, given $\tau > 0$, $k : [0, +\infty) \to \mathbb{R}$ and its Laplace transform K, a fractional quadrature based on a linear multistep methods to approximate convolution integrals reads

$$\int_0^t k(s) g(t-s) \mathrm{d}s \simeq \sum_{0 \leqslant j\tau \leqslant t} w_j g(t - j\tau), \quad 0 \leqslant t \leqslant T \tag{4}$$

for every locally integrable $g : [0, +\infty) \to \mathbb{R}$, where the weights w_j, for $j \geqslant 0$, are defined by means of the generating function

$$K\left(\frac{\delta(\xi)}{\tau}\right) = \sum_{j=0}^{+\infty} w_j \xi^j$$

being δ the quotient of the characteristic polynomials of the multistep method.

Convolution quadratures of type (4) to approximate convolution integrals in abstract frameworks has been deeply studied in literature. In fact, in [6,7] theses quadratures has been studied in the framework of kernels with sectorial Laplace transform, whose results were later extended for the derivatives of these integrals (see [8]).

Concerning to integro-differential equations of fractional order, numerical methods based on quadratures of type (4) have been studied to solve numerically equations of type (1) in the abstract setting of the sectorial operators, e.g. the ones based on the backward Euler method (see [3]) and BDF of two steps (see [1]). Moreover, for the second one the regularity required for the data to guarantee the optimal order of the method was also studied in [1].

☆ The author was supported by DGI-MCYT under Project MTM 2004-07194 cofinanced by FEDER funds. This paper was presented in the special session Fractional Differential Equations: Theory and Applications.

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doi:10.1016/j.advengsoft.2008.12.005

From the implementation point of view, it is very well know the high computational cost of these algorithms. In this way, the effort at present concentrates in the obtention of higher order methods as well as more efficient algorithms as, e.g. fast convolution quadratures (see [12]).

Keeping in mind some ideas of Lubich in [8], in this paper we study a numerical discretization for (3) of variable step size based on an adaptive fractional quadrature. In fact, the quadrature considered here is based on the backward Euler method of variable step size.

The paper is organized as follows. In Section 2 we introduce the adaptive quadrature proposed to define the numerical scheme for (3) which is describe in detail in Section 3. Section 4 is devoted to show the main result and its proof and the paper concludes with some numerical experiments shown in Section 5.

2. Adaptive quadratures of fractional type

Let g be a locally integrable function in $\mathbb{R}^+$ and let us consider the integral

$$\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s)\mathrm{d}s, \quad t \geqslant 0. \tag{5}$$

Notice that the Laplace transform of the kernel, which reads $\xi \mapsto \xi^{-\alpha}$, is analytic in the sector

$$S_{\pi(1-\alpha/2)} := \{\xi \in \mathbb{C} : |\arg(-\xi)| < \pi(1-\alpha/2)\}.$$

The key point of our approach consists in writing (5) in a different manner (see [7]), i.e.,

$$\begin{aligned}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s)\mathrm{d}s &= \int_0^t \left\{\frac{1}{2\pi i}\int_\gamma \mathrm{e}^{\xi(t-s)}\xi^{-\alpha}\mathrm{d}\xi\right\} g(s)\mathrm{d}s \\ &= \frac{1}{2\pi i}\int_\gamma \xi^{-\alpha} y(\xi,t)\mathrm{d}\xi,\end{aligned} \tag{6}$$

where γ stands for a suitable path connecting $-i\infty$ and $+i\infty$, with increasing imaginary part, lying outside of the sector $S_{\pi(1-\alpha/2)}$, and

$$y(\xi,t) = \int_0^t \mathrm{e}^{\xi(t-s)} g(s)\mathrm{d}s$$

represents the solution of the initial value problem

$$y' = \xi y + g, \quad y(0) = 0. \tag{7}$$

The idea of the fractional quadratures is based on the application of classical methods in (7) replacing later y by the numerical solution. In fact, recalling that the backward Euler method applied to (7) reads

$$y_n = \tau \sum_{j=1}^n r(\tau\xi)^{n-j} g(t_j), \quad n \geqslant 1, \tag{8}$$

where $y_n \simeq y(t_n)$ and $r(z) = 1/(1-z)$ represents the quotient of the characteristic polynomials, the fractional quadrature based on the backward Euler method is obtained replacing (8) in (6), i.e.,

$$\begin{aligned}\int_0^{t_n} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s)\mathrm{d}s &\simeq \frac{\tau}{2\pi i}\int_\gamma \xi^{-\alpha}\sum_{j=1}^n r(\tau\xi)^{n-j} g(t_j)\mathrm{d}\xi \\ &= \tau\sum_{j=1}^n w_{n-j}^{(\alpha)} g(t_j)\end{aligned} \tag{9}$$

being

$$w_j^{(\alpha)} := \frac{\tau}{2\pi i}\int_\gamma \xi^{-\alpha} r(\tau\xi)^j \mathrm{d}\xi, \quad 1 \leqslant j \leqslant n. \tag{10}$$

Our goal consists in getting an extension of these ideas for a variable step size backward Euler method. To this end let us consider the step setting

$$\begin{aligned}&0 = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = T, \\ &\tau_n := t_n - t_{n-1} \quad \text{and} \quad H := \max \tau_j\end{aligned} \tag{11}$$

and let assume that there exists $0 < \Omega < 1$ such that

$$1/\Omega \leqslant \frac{\tau_i}{\tau_j} \leqslant \Omega.$$

The backward Euler method with variable step size is now written as

$$y_n = \sum_{j=1}^n \tau_j \prod_{p=j}^n r(\tau_p\xi) g(t_j)$$

and following the same ideas as for the scheme with constant step size, the quadrature we propose reads

$$\int_0^{t_n} \frac{(t_n-s)^{\alpha-1}}{\Gamma(\alpha)} g(s)\mathrm{d}s \simeq \sum_{j=1}^n w_{j,n}^{(\alpha)} g(t_j), \tag{12}$$

where

$$w_{j,n}^{(\alpha)} := \frac{\tau_j}{2\pi i}\int_\gamma \xi^{-\alpha}\prod_{p=j}^n r(\tau_p\xi)\mathrm{d}\xi, \quad 1 \leqslant j \leqslant n. \tag{13}$$

Notice that, if constant step size is considered, both quadratures (4) and (12) coincide.

On the other hand, since the quadrature (12) and (13) does not preserve the convolution structure of the fractional integral on the contrary as occurs for the convolution quadrature (9) and (10), the ideas in [1–3,8] are not longer valid in our proofs. Therefore, a different approach will be considered in the present framework.

3. Numerical scheme

The Eq. (2) might suggest different treatments for the two integrals involved but in our framework the same treatment is allowed for both of them. However, for the integral $\int_0^t f(s)\,\mathrm{d}s$, the regularity of f plays an important role in the consistence on the numerical scheme. To overcome this difficulty, getting a cleaver presentation of our results, and without loss of generality, in the present paper we focus on the homogeneous equation, i.e., $f \equiv 0$, which will be interesting enough to illustrate the main ideas of our work.

In fact, let us consider

$$u(t) = u_0 + \partial^{-\alpha}\lambda u(t), \quad t \geqslant 0 \tag{14}$$

for $1 < \alpha < 2$ and λ lying inside of the sector $S_{\pi(1-\alpha/2)}$. This assumption on λ allows us to guarantee that (14) is well posed.

Notice that, a naive implementation of the quadrature (12) and (13) in Eq. (14) leads to the numerical scheme

$$u_n = u_0 + \lambda\sum_{j=1}^n w_{j,n}^{(\alpha)} u_j, \quad n \geqslant 1. \tag{15}$$

From a practical point of view, (15) directly yields the numerical algorithm, however in the theoretical framework, the proof of the convergence does not seem to be easy.

In this way we propose a sightly different approach which begins by taking the Laplace transform in both sides of (14). In fact, we have

$$U(\xi) = \frac{u_0}{\xi} + \frac{\lambda U(\xi)}{\xi^\alpha},$$

where U stands for the Laplace transform of u. Thus,

$$U(\xi) = \frac{\xi^{\alpha-1}}{\xi^\alpha - \lambda} u_0$$