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Certified computation of planar Morse–Smale complexes

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A R T I C L E   I N F O

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A B S T R A C T

The Morse–Smale complex is an important tool for global topological analysis in various problems of computational geometry and topology. Algorithms for Morse–Smale complexes have been presented in case of piecewise linear manifolds (Edelsbrunner et al., 2003a). However, previous research in this field is incomplete in the case of smooth functions. In the current paper we address the following question: Given an arbitrarily complex Morse–Smale system on a planar domain, is it possible to compute its certified (topologically correct) Morse–Smale complex? Towards this, we develop an algorithm using interval arithmetic to compute certified critical points and separatrices forming the Morse–Smale complexes of smooth functions on bounded planar domain. Our algorithm can also compute geometrically close Morse–Smale complexes.

1. Introduction

Geometrical shapes occurring in the real world are often extremely complex. To analyze them, one associates a sufficiently smooth scalar field with the shape, e.g., a density function or a function interpolating gray values. Using this function, topological and geometrical information about the shape may be extracted, e.g., by computing its *Morse–Smale complex*. The cells of this complex are maximal

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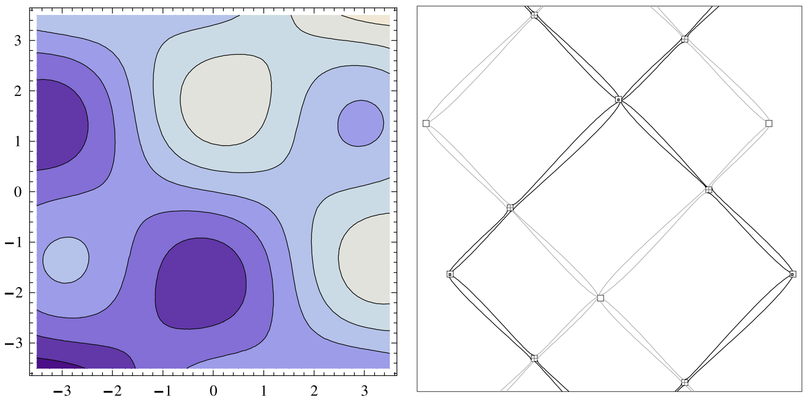
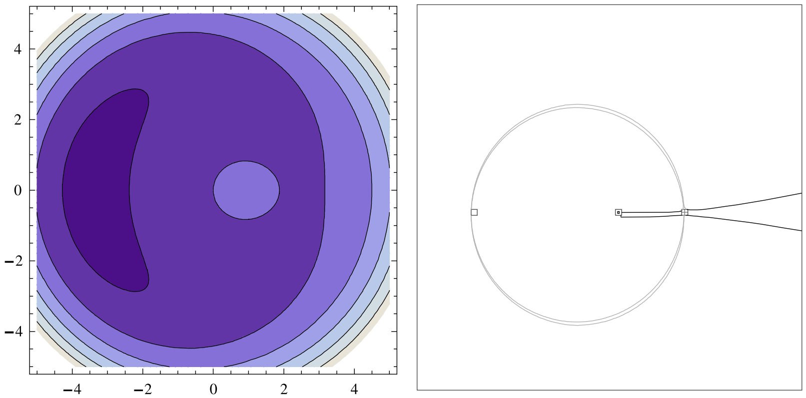


(a) Contour plot (left) and Morse-Smale complex (right) of $h(x, y) = \cos x \ \sin y + 0.2\,(x + y)$ inside box $[-3.5, 3.5] \times [-3.5, 3.5]$. CPU-time: 11 seconds.



(b) Contour plot (left) and Morse-Smale Complex (right) of $h(x, y) = 10x - \frac{13}{2}(x^2 + y^2) + \frac{1}{3}\,(x^2 + y^2)^2$ inside box $[-5, 5] \times [-5, 5]$. CPU-time: 0.5 seconds.

Fig. 1. Contour plots of Morse–Smale functions, and their Morse–Smale complexes.

connected sets consisting of orthogonal trajectories of the contour lines—curves of steepest ascent—with the same critical point of the function as origin and the same critical point as destination. The leftmost plots in Figs. 1(a) and 1(b) illustrate the level sets of such a density function h, and the rightmost pictures the Morse–Smale complex of h as computed by the algorithm in this paper. This complex reveals the global topology of the shape. Recently, the Morse–Smale complex has been successfully applied in different areas like molecular shape analysis, image analysis, data and vector field simplification, visualization and detection of voids and clusters in galaxy distributions (Cazals et al., 2003; Gyulassy et al., 2008).

1.1. *Problem statement*

A Morse function $h : \mathbb{R}^2 \to \mathbb{R}$ is a real-valued function with *non-degenerate critical points* (i.e., critical points with non-singular Hessian matrix). Non-degenerate critical points are *isolated*, and are either maxima, or minima, or saddle points. They correspond to singular points of the *gradient vector field* ∇h of h, of type sink, source or saddle, respectively. Regular integral curves of the gradient vector field ∇h are orthogonal trajectories of the regular level curves of h. We are interested in the configuration of integral curves of the gradient vector field. An *unstable (stable) separatrix* of a saddle point is the set of all regular points whose forward (backward) integral curve emerges from the