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MICC: A tool for computing short distances in the curve complex

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The complex of curves $\mathcal{C}(S_g)$ of a closed orientable surface of genus $g \geq 2$ is the simplicial complex whose vertices, $\mathcal{C}^0(S_g)$, are isotopy classes of essential simple closed curves in S_g. Two vertices co-bound an edge of the 1-skeleton, $\mathcal{C}^1(S_g)$, if there are disjoint representatives in S_g. A metric is obtained on $\mathcal{C}^0(S_g)$ by assigning unit length to each edge of $\mathcal{C}^1(S_g)$. Thus, the distance between two vertices, $d(v, w)$, corresponds to the length of a geodesic—a shortest edge-path between v and w in $\mathcal{C}^1(S_g)$. In Birman et al. (2016), the authors introduced the concept of *efficient geodesics* in $\mathcal{C}^1(S_g)$ and used them to give a new algorithm for computing the distance between vertices. In this note, we introduce the software package MICC (*Metric in the Curve Complex*), a partial implementation of the efficient geodesic algorithm. We discuss the mathematics underlying MICC and give applications. In particular, up to an action of an element of the mapping class group, we give a calculation which produces all distance 4 vertex pairs for $g = 2$ that intersect 12 times, the minimal number of intersections needed for this distance and genus.

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1. Introduction

Let S or S_g denote a compact, connected, orientable surface of genus g, where $g \geq 2$. A simple closed curve on S is *essential* if does not bound a disk in S. The complex of curves, introduced by Harvey (1981), is the simplicial complex, $\mathcal{C}(S)$, whose vertices (or 0-skeleton), $\mathcal{C}^0(S)$, are isotopy classes of essential simple closed curves; and, whose edges of the 1-skeleton, $\mathcal{C}^1(S)$, connect vertices that have disjoint representatives. For the remainder of this note, "curve" will mean "simple closed curve". By declaring that each edge of $\mathcal{C}^1(S)$ has length 1, we endow $\mathcal{C}^0(S)$ with a metric. Specifically, an *edge path* is a sequence of vertices $\{v = v_0, v_1, \cdots, v_n = w\}$ such that $d(v_i, v_{i+1}) = 1$. A *geodesic path* joining v and w is a shortest edge-path. The *distance*, $d(v, w)$, between arbitrary vertices is the length of a geodesic path. Since it is known that the complex of curves is connected, which was stated by Harvey (1981) and followed from a previous argument of Lickorish (1964), the value $d(v, w)$ is well-defined for all vertex pairs.

The coarse geometric properties of the complex of curves were first studied extensively by Masur and Minsky (2000; 1999). One of the premier results of their work is that the $\mathcal{C}(S)$ is δ*-hyperbolic*—geodesic triangles in $\mathcal{C}^1(S)$ are δ-thin in the sense that any one edge is contained in the δ-neighborhood of the union of the other two edges. The complex of curves has proved a useful tool for the study of hyperbolic 3-manifolds, mapping class groups and Teichmüller theory. Indeed, Masur and Minsky establish that Teichmüller space (with the Teichmüller metric) can be enlarged to an *electric Teichmüller space* that is quasi-isometric to the $\mathcal{C}^1(S)$ (Masur and Minsky, 2000, cf. Theorem 1.2 & Lemma 3.1). The concept of enlarging a geodesic metric space to an "electric space" relative to a collection of regions in the metric space is due to Farb (1998). In an analogous fashion to Teichmüller space, Masur and Minsky prove that the mapping class group, *Mod*(S), or more strictly speaking, its Cayley graph, can be enlarged to an electric space that is also quasi-isometric to $\mathcal{C}^1(S)$ (Masur and Minsky, 2000, cf. Theorem 1.3 & Lemma 3.2). Thus, both Teichmüller space (with the Teichmüller metric) of S and *Mod*(S) are *relatively hyperbolic*. Finally, it has been established that there exists a δ that is independent of genus g, i.e. we have *uniform hyperbolicity* (Aougab, 2013; Bowditch, 2014; Clay et al., 2013; Hensel et al., 2015).

Specifically regarding efforts at making explicit distance calculations, Leasure initially proved the existence of an algorithm to compute the distance between two vertices of $\mathcal{C}^0(S)$ (Leasure, 2002, cf. Corollary 3.2.6). Later, other algorithms that utilize properties of the coarse geometry were established by Shackleton (2012) and Webb (2013). However, none of these algorithms can reasonably be implemented due to the large amount of case counting needed even for short distances and small genus.

There are two types of local pathology in $\mathcal{C}^1(S)$ that make calculating distance for specific vertex pair problematic. First, $\mathcal{C}^1(S)$ is locally infinite, since there are infinitely many essential curves representing distinct isotopy classes disjoint from any fixed similar curve; this means that there are infinitely many vertices that are distance 1 from any given vertex. Second, there are typically infinitely many distinct geodesics joining any two vertices. To take a simple illustration of this second pathology, consider two curves, $\alpha, \beta \subset S_2$, that intersect exactly once. The complement of $\alpha \cup \beta$ is homeomorphic to a once punctured genus one surface. In this genus one sub-surface of S_2 there are infinitely many essential curves representing distinct vertices of $\mathcal{C}^0(S_2)$. Thus, the vertices associated with α and β are distance two apart, *and* there are infinitely many distinct geodesics between their corresponding vertices. Although there are straightforward procedures for constructing an edge path in $\mathcal{C}^1(S_2)$ between two arbitrary vertices, there are infinitely many such edge paths (due to the first pathology) and infinitely many geodesic paths (due to the second pathology), making identifying a shortest edge path for computing distance challenging.

Recently Birman, Margalit and the second author (Birman et al., 2016) have given a new algorithm—*the efficient geodesic algorithm*—and we have developed an implementation of it called the *Metric in the Curve Complex* (MICC). Applications of MICC we will present in this note include:

(i) establishing that the minimal geometric intersection number for vertices of $\mathcal{C}(S_2)$ with distance four is 12,