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Hybrid time-space dynamical systems of growth bacteria with applications in segmentation

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ABSTRACT

A biological dynamic system carries engineering properties such as control systems and signal processing (or image processing) addicted to molecular biology at the level of bio-molecular communication networks. Dynamical system features and signal reply functions of cellular signaling pathways are some of the main topics in biological dynamic systems (for example the biological segmentation). In the present paper, we introduce new generalized hybrid time-space dynamical systems of growing bacteria. We impose the approximate analytic solution for the system. The generalization adapted the concepts of the Riemann–Liouville fractional operators for time and the Srivastava–Owa fractional operators for space. Moreover, we introduce a numerical perturbation method of two operators to obtain the approximate solutions. We establish the existence and uniqueness results and impose some applications in the sequel. Moreover, we study the Ulam stability and apply these stable solutions to improve the segmentation of a class of growing bacteria.

1. Introduction

Fractional derivatives (real and complex) (see [1–3]) can definite the properties of memory and heredity of supplies. Applied problems involve characterizations of fractional derivatives letting the usage of initial conditions. Fractional time derivatives are associated with fractional sub-diffusion, where particles feast more slowly than a classical diffusion. While the fractional space derivatives are utilized to model fractional diffusion or distribution, where particles feast at a rate not in agreement with the classical Brownian motion model [4]. The biological models are characterized as discrete, continuum (time - space), or hybrid, which contain both discrete and continuum components. Discrete (e.g. agent- based or cellular automata) models characterize individual components repeatedly (e.g. cells), and can simply join biological rules for interactions or transitions (e.g. chemotaxis or haptotaxis). Continuum models describe the continuous distribution of supplies across the area (e.g. a tumor as a continuous medium of cells). While cells are reflex objects, this modeling background is sensible when seeing large populations of cells, and is particularly advantageous when involving chemical kinetics and diffusion substances. Hybrid models combine the two methods, though typically with additional numerical contests to join the discrete to the continuous illustrations. An early sample of a hybrid model is established in [5], in which antigenic growth of capillaries from sprout tips is captured by a discrete biased random-walk type, whereas the responses of cells are represented as coupled reaction-diffusion equations. In general, a mathematical structure together with an interpretive rule is the main aim of the study. If the interpretive rule is missing the equations cannot be considered as a model and cannot tell anything about biology. One of these rules is by using the concept of the segmentation. In this paper, we introduce a new method of segmentation based on the solution of the dynamic system.

In computer visualization, the image segmentation is the procedure of dividing a digital image into multiple segments (sets of pixels). The purpose of segmentation is to simplify, modify and characterize the image into approximate image that is more expressive for analysis [6]. The segmentation usage is to realize

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boundaries and pieces such as lines and curves in images. More precisely, the image segmentation is the technique of transmission a label to every pixel in an image such that pixels with the same label segment certain features. The outcome of image segmentation is a set of segments that jointly cover the entire image, or a set of contours removed from the image. Fractional image segmentation has been studied in [7,8], by applying some fractional formal such as the fractional non-Markov Poisson stochastic process and fractional sinc function respectively.

Our aim is to introduce new generalized hybrid time-space dynamical systems of growing bacteria. The generalization adapted the concepts of Riemann–Liouville fractional operators for time and Srivastava–Owa fractional operators for space. Moreover, we give a numerical perturbation method of the two operators to obtain the approximate solutions. We establish the existence and uniqueness results with applications. Also, we investigate the Ulam stability. Finally, by using these applications, we improve the segmentation of a class of growing bacteria (populations of cells). The proposed method shows high accuracy of segmentation comparing with formerly works.

2. Processing methods

In complex analysis, the Srivastava and Owa fractional operators are defined in z-plane $\mathbb{C}$ as follows (see [1]):

Definition 2.1. The non integer (fractional) derivative of order β is known, for a function $f(z)$ in the expression

$$D_z^\beta f(z) := \frac{1}{\Gamma(1-\beta)}\frac{d}{dz}\int_0^z \frac{f(\zeta)}{(z-\zeta)^\beta}d\zeta; \qquad 0 \le \beta < 1,$$

where the function $f(z)$ is analytic in simply-connected region belonging to $\mathbb{C}$ and containing the origin with the multiplicity of $(z-\zeta)^{-\beta}$. The corresponding fractional integral of order β is formulated by

$$I_z^\beta f(z) := \frac{1}{\Gamma(\beta)}\int_0^z f(\zeta)(z-\zeta)^{\beta-1}d\zeta; \qquad \beta > 0.$$

The above calculus implies:

$$D_z^\beta z^\mu = \frac{\Gamma(\beta+1)}{\Gamma(\mu-\beta+1)}z^{\mu-\beta}, \;\; \mu > -1; \qquad 0 \le \beta < 1$$

and

$$I_z^\beta z^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\beta+1)}z^{\mu+\beta}, \;\; \mu > -1; \qquad \beta > 0.$$

In our investigation, the variable $z = x + iy$ corresponds to the two dimensional space. While the time corresponds to the variable t. In this case, we need the Riemann–Liouville operators ([2]).

Definition 2.2. The arbitrary order integral $(\alpha > 0)$ of the function f is expressed by

$$I_a^\alpha f(t) = \int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}f(\tau)d\tau.$$

The arbitrary order derivative $(0 \le \alpha < 1)$ of the function f is defined by

$$D_a^\alpha f(t) = \frac{d}{dt}\int_a^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)}f(\tau)d\tau = \frac{d}{dt}I_a^{1-\alpha}f(t).$$

The operators are achieved the following property:

$$D^\alpha t^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}t^{\mu-\alpha}, \;\; \mu > -1; \;\; 0 < \alpha < 1$$

and

$$I^\alpha t^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}t^{\mu+\alpha}, \;\; \mu > -1; \;\; \alpha > 0.$$

In this article, we consider the time-space fractional differential equation (Cauchy type) of the form

$$D^\alpha D_z^\beta u(t,z) = f(t,z,u) \tag{1}$$

where $u(0,0) = 0$ in a neighborhood of the origin, $t \in J := [0,T]$, $z \in U := \{z \in \mathbb{C} : |z| < 1\}$, $u : J \times U \to \mathbb{C}$ and $f : J \times U \times \mathbb{C}$ is analytic function. Eq. (1) is a generalization of the population systems in [9].

The most significant benefit of utilizing any class of fractional differential equations in mathematical modelling is their non-local property (see [10–12]). Obviously, the integer order differential operator is a local operator while the fractional order differential operator is non-local. This leads to the next formal of a system depends not only upon its present state but also upon all of its historical statuses. These advantages make the field of fractional more and more popular in scientific and technological areas. Recently, different procedures employed to handle numerous biological problems. One of these methods is the homotopy technique. The homotopy analysis technique occurred for solving linear and nonlinear not only differential equations, but also integral equations. Diverse from perturbation method, the homotopy analysis technique does not request any small or large parameters in the equations (see [13]).

Our aim is to study the existence and uniqueness of Eq. (1). Furthermore, we introduce the homotopy perturbation method to get the analytic solution in a complex domain. We check our results by using the Ulam stability of fractional order. We simulate generalized cases, by illustrating numerical examples. A comparison between solutions is given to these special cases with computation of errors. We use these solutions, in term of the Mittag–Leffler function to improve the segmentation of images containing growing bacteria.

3. Findings

Let $J := [0,T]$, $\triangle := J \times U$, $\mathbb{B} := CHB[\triangle \times \mathbb{C}, \mathbb{C}]$ be a complex Banach space of all continuous, analytic and bounded functions in $\triangle$ endow with the max norm. We have the following result:

Theorem 3.1. *Assume that $f \in \mathbb{B}$ is a Lipschitz function in $u \in CH(\triangle, \mathbb{C})$ (the space of continuous in J and analytic in U). Then (1) has a unique solution in a neighborhood of the origin (0,0) $\in \triangle$ provided*

$$\frac{LT^\alpha}{\Gamma(\beta+1)\Gamma(\alpha+1)} < 1, \quad L \in (0,\infty).$$

Proof. Define the operator

$$(Pu)(t,z) := \frac{1}{\Gamma(\beta)}\int_0^z (z-\zeta)^{\beta-1}\left[\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}f(\tau,\zeta,u)d\tau\right]d\zeta.$$

Denotes the norm of the function f by $\|f\|_{\mathbb{B}} = \max_{(t,z)\in\triangle}|f(t,z,u)| = M < \infty$. First, we show that P is bounded in a neighborhood of the origin (0,0) $\in \triangle$

$$\begin{aligned}|(Pu)(t,z)| &= \left|\frac{1}{\Gamma(\beta)}\int_0^z (z-\zeta)^{\beta-1}\left[\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}f(\tau,\zeta,u)d\tau\right]d\zeta\right| \\ &\le \left|\frac{1}{\Gamma(\beta)}\int_0^z (z-\zeta)^{\beta-1}\left[\frac{MT^\alpha}{\Gamma(\alpha+1)}\right]d\zeta\right| \\ &\le \frac{MT^\alpha}{\Gamma(\beta+1)\Gamma(\alpha+1)}.\end{aligned}$$

We proceed to prove that P is a contracting mapping. By the assumption we conclude that

$$|(Pu)(t,z) - (Pv)(t,z)|$$