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Momentum balance in the shallow water equations on bottom discontinuities

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ABSTRACT

This work investigates the topical problem of balancing the shallow water equations over bottom steps of different heights. The current approaches in the literature are essentially based on mathematical analysis of the hyperbolic system of balance equations and take into account the relevant progresses in treating the non-conservative form of the governing system in the framework of path-conservative schemes. An important problem under debate is the correct position of the momentum balance closure when the bottom elevation is discontinuous. Cases of technical interest are systematically analysed, consisting of backward-facing steps and forward-facing steps, tackled supercritical and subcritical flows; critical (sonic) conditions are also analysed and discussed.

The fundamental concept governing the problem and supported by the present computations is that the energy-conserving approach is the only approach that is consistent with the classical shallow water equations formulated with geometrical source terms and that the momentum balance is properly closed if a proper choice of a conventional depth *on* the bottom step is performed. The depth on the step is shown to be included between the depths just upstream and just downstream of the step. It is also shown that current choices (as given in the literature) of the depth on (or in front of) the step can lead to unphysical configurations, similar to some energy-increasing solutions.

1. Introduction

The numerical integration of shallow water equations (SWE) with source terms has been intensively investigated in recent years due to the significant advances in computational fluid dynamics in finite difference methods, finite volume methods, discontinuous Galerkin methods, and so forth (Caleffi and Valiani, 2009, 2012, Caleffi et al., 2006, 2016). Particular interest has been devoted to discontinuous solutions, typically due to the physics of stationary jumps and moving bores generated by the hyperbolic nature of the homogeneous problem, in light of the more general Rankine–Hugoniot theory of inviscid shock propagation. A further source of discontinuity is due to the possible discontinuous profile of the bottom elevation, which introduces singularities in the term related to the topography.

From recent works on proper numerical methods for SWE on discontinuous bed elevations, the first contribution may be considered (Alcrudo and Benkhaldoun, 2001). The authors analysed the similarity solutions of the Riemann problem over a step, evidencing the existence of a standing discontinuity over the step and studying a relevant number of possible solutions. They imposed total head conservation across the step and highlighted the need for an additional kinetic energy term to properly take the energy dissipation into account, when necessary. This work has the merit of highlighting the wide variety of solutions that can be found, which are considerably more numerous than the classical case of the Riemann problem over a flat horizontal bottom. Moreover, this work demonstrates the potential complexity of any numerical method that has the ambition to incorporate the variability of such a number of situations.

The study LeFloch and Thanh (2011), which is a deepening of the previous study LeFloch and Thanh (2007), investigated the Riemann problem over a discontinuous bed elevation and provided a detailed analysis of the possible solutions. This work is likely the most complete study on this topic. More specifically, the authors investigated the existence and uniqueness conditions for such solutions, showing the uniqueness in the non-resonant regime and the existence of multiple solutions in the resonant regime. The eigensystem analysis provides two genuinely hyperbolic characteristic fields and one linearly degenerate field. Regarding the linearly degenerate field, it is shown that across a discontinuity, the bottom

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elevation must remain constant or the discontinuity must be stationary. This result means that over a discontinuous bed, the propagation celerity must be zero. The jump relations on the discontinuity provide the constancy of the unit discharge and the conservation of the total head of the flow, which are exactly the assumptions adopted herein.

The investigation is extended in Han and Warnecke (2014), where the exact Riemann solutions for the shallow water equations with a bottom step is completed, including the dry bed problem, which can occur in specific circumstances.

The conservation of the discharge and of the total head are successfully applied in 1D modelling of open-channel hydraulics, both for the unit width channel and in a channel of arbitrary shape (Murillo and Garcia-Navarro, 2013, 2014; Navas-Montilla and Murillo, 2015).

A different approach is presented in Bernetti et al. (2008), which presents the exact solution of the Riemann problem for shallow water equations with a step-like bottom. The solution was obtained by solving the system that included an additional equation for the bottom geometry and then using the principles of conservation of mass and momentum across the step. By writing the Rankine–Hugoniot condition for the standing wave across the step, it is shown that the resulting solution is unique and satisfies the principle of the dissipation of energy across the shock wave. Because all the methods establish the momentum integral principle across the step, the key step is estimating the force exerted by the step on the fluid. In the framework of SWE and therefore assuming a hydrostatic distribution of pressure, it is possible to express the step force as a function of an equivalent, conventional depth *on* the step. In the case of a forward-facing step, such a depth is evaluated, as in all the other similar methods, using the depth just in front of the step measured from the centroid of the step. In the case of a backward-facing step, which is not explicitly described, it can be argued that the depth must be evaluated again as the depth just in front of the step measured from the centroid of the step. More specifically, the phrase “in front of” means the upstream flow depth minus half the step height in the forward-facing step case, and it means the downstream flow depth minus half the step height depth in the backward-facing step case. This choice is similar to the other methods described in the following (Cozzolino et al., 2011; Rosatti and Begnudelli, 2010; Rosatti and Fraccarollo, 2006; Rosatti et al., 2008), even if the formal position of the momentum balance appears slightly different.

A strictly closed approach to the last one is that from Rosatti and Fraccarollo (2006), which investigated the numerical computation of one-dimensional, unsteady, free surface flows over a mobile bed. They considered a strong interaction between the flow and the erodible bottom, taking non-conservative terms in the momentum equation into account and solving the related Riemann problem. The technique was named AWB (approximately well balanced), and it was applied to schematic and experimental test cases. At the simplest order of approximation, they considered a backward-facing step as the typical scheme concerning the flow across two consecutive computational cells. The force exerted by the step on the fluid is proportional to the height of the step and to a “proper” water depth “in front of” the step.

The word “proper” means that the depth must be measured from the centroid of the step. The term “in front of” means that the downstream (or right) depth must be considered if the step is facing backward and that the upstream (or left) depth must be considered if the step is facing forward. The force is clearly exerted on the fluid in the flow direction in the former case, whereas it is opposite to the flow direction in the latter case. The context of this work is broader than that of the present work because (as in Rosatti et al. (2008)) a mobile bed is considered, but the same procedure was also used for a fixed bed elevation (Rosatti and Begnudelli, 2010).

In particular, Rosatti and Begnudelli (2010) thoroughly analysed the Riemann problem for the one-dimensional shallow water equations from theoretical and numerical perspectives. The analysis of the wave at the step leads to a non-conservative crossing of the step in terms of total head. The momentum balance on the step is closed using an integral momentum balance that is very similar to that proposed herein. The only difference is the estimate of the depth on the step, which is performed in Rosatti and Begnudelli (2010) using the same technique proposed by Rosatti and Fraccarollo (2006) and Bernetti et al. (2008). However, the simple reasoning proposed here does not require a new formulation of the Riemann invariants, and a proper choice of the special depth satisfies both momentum and energy conservation requirements.

A recent contribution to the problem which given by Cozzolino et al. (2011), which analysed the shallow water equations on bottom discontinuities. In this work, the hydrostatic-like pressure distribution at the step, according the concepts of Bernetti et al. (2008) and Rosatti and Fraccarollo (2006), is also analysed and discussed in the framework of the path-conservative theory of Maso et al. (1995). They rebut the preservation of the total head through the bed step (Alcrudo and Benkhaldoun, 2001; Caleffi and Valiani, 2009; Caleffi et al., 2016; LeFloch and Thanh, 2011), claiming that experimental evidence shows that the dissipation on the step is a well-known phenomenon. They consider that the integral momentum balance, taking into account the force that the bed exerts onto the flow, conflicts with the total head conservation hypothesis. In Cozzolino et al. (2011), they explicitly make reference to the flow detachment and reattachment and eddy recirculation cells, which subtract mechanical energy from the mean flow.

By contrast, the hypothesis supported here is that the dissipation is evident, but it is not automatically incorporated in the shallow water scheme. If a proper source term is not added, then the total head is conserved, and integral momentum conservation can also be preserved by simply selecting a proper depth on the step.

Caleffi and Valiani (2009) proposed a finite volume WENO scheme, which is fourth-order accurate in space and time, for the numerical integration of shallow water equations with the bottom slope source term. The method for managing bed discontinuities is based on a suitable reconstruction of the conservative variables at the cell interfaces, coupled with a correction of the numerical flux based on the local conservation of total energy. Properly selected test cases show the efficiency of the method in treating discontinuities. The method finds a flux correction on bed steps that takes into account the force exchanged between the step itself and the fluid, and the flux balancing is achieved not only in still water conditions but, more generally, also for each steady-state dynamic condition. The analytical treatment of the step that is proposed here is conceived as a deepening of the force estimate on the step itself.

In Caleffi et al. (2016), using an unified framework consisting of a third-order accurate discontinuous Galerkin scheme, five different numerical methods applied to the free-surface shallow flow simulation on bottom steps are compared. The role that the treatment of bottom discontinuities plays in the preservation of specific asymptotic conditions is examined. In particular, three widespread approaches based on the motionless steady state balancing (Audusse et al., 2004; Kesserwani and Liang, 2011; Parés, 2006) are compared with two approaches (Caleffi and Valiani, 2009; Caleffi et al., 2016) that are based on the preservation of a moving-water steady state. The fundamental findings support the concept that the well balancing of a moving steady state (rather than a well-balanced model in the case of still water) significantly improves the overall behaviour of the schemes. This perspective is successful both in the framework of the classical finite volume