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Explicit wave action conservation for water waves on vertically sheared flows

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ABSTRACT

This paper addresses a major shortcoming of the current generation of wave models, namely their inability to describe wave propagation upon ambient currents with vertical shear. The wave action conservation equation (WAE) for linear waves propagating in horizontally inhomogeneous vertically-sheared currents is derived following Voronovich (1976). The resulting WAE specifies conservation of a certain depth-averaged quantity, the wave action, a product of the wave amplitude squared, eigenfunctions and functions of the eigenvalues of the boundary value problem for water waves upon a vertically sheared current. The formulation of the WAE is made explicit using known asymptotic solutions of the boundary value problem which exploit the smallness of the current magnitude compared to the wave phase velocity and/or its vertical shear and curvature; the adopted approximations are shown to be sufficient for most of the conceivable applications. In the limit of vanishing current shear, the new formulation reduces to that of Bretherton and Garrett (1968) without shear and the invariant is calculated with the current magnitude taken at the free surface. It is shown that in realistic oceanic conditions, the neglect of the vertical structure of the currents in wave modelling which is currently universal might lead to significant errors in wave amplitude. The new WAE which takes into account the vertical shear can be better coupled to modern circulation models which resolve the three-dimensional structure of the uppermost layer of the ocean.

1. Introduction

In nature, wind waves and swell almost always propagate on vertically sheared currents in a horizontally inhomogeneous environment. Due to interaction with the atmosphere, ocean currents of any origin usually have a boundary layer in the uppermost layer of the ocean, the layer where most of the surface wave motion is localized. In recent years ocean circulation models have been significantly improved, especially for modelling relatively small areas, often coastal, where most of the offshore activities and shipping lanes are concentrated, and now have the capability to describe dynamics of vertically-sheared currents with an increasingly fine vertical and horizontal resolution, e.g. Bellafiore and Umgiesser (2010). However, all wave models employed in commercial wave forecasting today still only take into account vertically-averaged mean flows which, as shown below, might lead to significant errors in realistic conditions. Minimizing such errors is important for a variety of engineering applications, e.g., for calculating the loads and impact on off-shore structures, sediment transport, etc. Also, since waves, to a large extent, control the exchange of momentum, heat and mass exchange between the ocean and atmosphere, capturing more accurately their dynamics and their coupling with currents is also a way towards improvement of the weather prediction and climate models (Cavaleri et al., 2012).

The primary goal of this work is to put forward an “explicit” closed form of the wave action conservation equation (WAE) suitable for operational forecasting which takes into account the vertical shear of the ambient currents. The seminal work of Bretherton and Garrett (1968) examined linear wavetrains in a moving media and deduced that it is the adiabatic invariant, which they called *wave action* (not the wave energy) that is conserved. They applied their fundamental idea of the wave action conservation to a large variety of waves, such as, e.g., sound waves, Alfvèn waves, internal gravity waves and inertial waves, Rossby waves, etc (Bretherton and Garrett, 1968). For the problem focused upon here, i.e. surface gravity waves propagating on currents, the co-existence of motions of vastly different scales in natural water basins presents a serious challenge for their direct numerical

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modeling. On the other hand, for water waves on currents in nature, the almost universal vast separation of spatial and temporal scales provides a possibility for developing an asymptotic description of the coupled evolution of waves and currents. To the leading order, the wave dynamics is captured by *fast* and *short* linear waves, while the evolution of the currents is devoid of fast and short scales. In the present work we focus entirely on the dynamics of such wave fields. Thus, we are interested in the phase averaged evolution of linear wave fields on the time and space scales shorter than those where nonlinear interactions become important. This range of scales is quite substantial. For example, for dominant wind waves this range is typically up to tens of minutes and kilometers, for swell it is up to tens of kilometers. At these scales inhomogeneities due to currents and topography are the dominant factors.

Here we assume an arbitrary current profile with non-uniform vorticity and exploit the scale separation. To this end, the WKBJ approach is employed following the mostly overlooked work by Voronovich (1976) (hereafter V76), where the WAE for linear surface and internal gravity waves on shear flows in a fluid of arbitrary depth was first derived. The significance of this work goes beyond the mere derivation of the WAE for the generic situation; it also demonstrated the role of the equations for low-frequency larger scale motions and, hence, the factors which might have negligible direct effect on waves, but are of importance for the mean flows and through this back door, on wave action conservation. Independently, White (1999) arrived at a similar derivation of the WAE but confined to deep water waves only. Both derivations yield an equation governing slow evolution of wave amplitude in space and time in an implicit form. To use it the eigenvalues (frequencies) and eigenfunctions (vertical modes) have to be found in each point of a wavepacket trajectory in “slow” space, which requires solving the boundary value problem for waves on a vertically-sheared current (the Rayleigh equation and the appropriate boundary conditions on the free surface and bottom). Then the solutions of the boundary value problem have to be used to find the packet trajectory and substituted into the WAE. Exact analytical solutions of the boundary value problem for an arbitrary current are not known. Probably it is this impediment which prevented the adoption of Voronovich’s findings in practical wave modelling. Here Voronovich’s derivation is revisited highlighting the junctions in the derivation where taking into account some extra effects such as the Earth’s rotation, ambient flow turbulence, wind forcing, etc., might also be important and result in a different WAE. *A priori* one could not rule out a noticeable effect of the Earth’s rotation despite the significant scale separation, since the numerical simulations of turbulent Reynolds’ stresses in sheared flow beneath the free surface show a significant effect caused by the rotation under the comparable scale separation (Zikanov et al., 2003). In the present work, Earth’s rotation is taken into account, while ambient flow turbulence and wind forcing are neglected. Still, they were discussed in orderto outline where and how these effects might enter the problem.

Taking into account the presence of the vertical shear of the currents also substantially affects the nonlinear dynamics of the waves propagating on the currents. In particular, the wave’s vertical structure differs from that for potential waves (Simmen and Saffman, 1985; Abrashkin and Zen’kovich, 1990), the timescale of the Benjamin–Feir instability ($O((\mu^2\omega)^{-1})$, μ is wave steepness, ω is wave frequency) changes (e.g. Oikawa et al., 1987) and triad resonant interactions between pairs of surface harmonics and a vorticity wave, which are absent in vertically uniform flows, become possible (Zakharov and Shrira, 1990) on the timescale of $O((\mu\omega)^{-1})$.It should be noted that for short wind waves of typical wavelength ~0.1 m, the nonlinear interactions can happen quite quickly but in this study we focus on wavelengths in the range 10–100 m and on linear dynamics of water waves on horizontally and vertically varying currents; the nonlinear interactions, both triad and cubic only have to be considered when attempting to describe wave evolution on longer timescales.

Although for the boundary value problem for waves on a current with an arbitrary vertical profile exact analytical solutions have not been found. Fortunately, in typical oceanic conditions there are always natural small parameters which can be exploited to get asymptotic solutions for generic profiles. Stewart and Joy (1974) derived an approximate dispersion relation for deep water waves on a depth-dependent current as the leading order term in an asymptotic expansion, the current magnitude normalized by the wave phase velocity being the small parameter. This advance was followed by a finite-depth extension of this approach by Skop (1987). The second order term in this expansion was found by Kirby and Chen (1989). An alternative solution of the deep water boundary value problem in terms of a converging series was derived by Shrira (1993) by exploiting the presumed smallness of vorticity and more recent work includes analysis of the boundary value problem for a piecewise linear approximation (Zhang, 2005)

This paper brings together both lines of inquiry: the implicit WAE formulation of Voronovich (1976), hereafter V76 and asymptotic solutions of the boundary value problem for waves on a sheared current. The V76 formulation is exact within the framework of the linearised Euler equations and the WKBJ approximation. Here we choose an approximate solution to the boundary value problem, most appropriate in our context, which makes the WAE explicit and balances the accuracy and simplicity. Thus, for an arbitrary vertical profile of the current we put forward a formulation of the WAE suitable for operational forecasting with an explicit wave action invariant for waves on a slowly varying current and topography. The discrepancies between the predictions of the new WAE and that for the vertically averaged currents, on the one hand, and the “exact” V76, on the other, are examined. We show that for sample realistic situations the adopted approximation indeed works well. The situations, where the discrepancy with the results for vertically averaged currents is significant, are identified.

Without any pretence at drawing a comprehensive review it makes sense to outline other lines of enquiry on water waves on shear currents to provide the context for this study. Most of the efforts concentrated on theoretical studies. As far as the linear theory is concerned the reviews by Peregrine (1976) and Peregrine and Jonsson (1983) are still relevant today. Just a few developments having relevance to the current study have to be specially noted. Although the scale separation underpinning the universally adopted WKBJ approach practically always holds in the ocean, the caustics do occur. In the vicinity of a caustic the field evolution does not result in a singularity predicted by the ray theory but needs to be described by a special model equation. In the absence of vertical shear in the vicinity of a turning point the model equation is the standard Airy equation. For waves on a vertically sheared current the problem is more complicated but it has been solved by McKee (1974); 1977). An independent derivation of the WAE for unidirectional waves on a linearly sheared collinear current was carried out by Jonsson et al. (1978). Since for this case it is possible to introduce a potential for the wave motion and the boundary value problem is straightforward to solve, no further approximations are needed, which makes it a very attractive toy model. We are not aware of it being applied to the modelling of any real situation in the ocean. The popularity of considering the constant shear currents does not seem to wane even in the linear setting; a recent paper by Ellingson and Brevik (2014) provides an update. The mild-slope equation, widely-used in nearshore and coastal regions, very recently was extended to include the effects of a linearly-sheared current (Touboul et al., 2016).