Ocean Modelling 116 (2017) 1–15

Contents lists available at ScienceDirect

Ocean Modelling

journal homepage: www.elsevier.com/locate/ocemod

# High-order accurate finite-volume formulations for the pressure gradient force in layered ocean models

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ARTICLE INFO

*Article history:*
Received 3 March 2017
Revised 21 April 2017
Accepted 13 May 2017
Available online 19 May 2017

*Keywords:*
Ocean modelling
Pressure gradient force
Generalised coordinates
Isopycnal coordinates
Terrain-following coordinates

ABSTRACT

Discretisation of the horizontal pressure gradient force in layered ocean models is a challenging task, with non-trivial interactions between the thermodynamics of the fluid and the geometry of the layers often leading to numerical difficulties. We present two new finite-volume schemes for the pressure gradient operator designed to address these issues. In each case, the horizontal acceleration is computed as an integration of the *contact* pressure force that acts along the perimeter of an associated momentum control-volume. A pair of new schemes are developed by exploring different control-volume geometries. Non-linearities in the underlying equation-of-state definitions and thermodynamic profiles are treated using a high-order accurate numerical integration framework, designed to preserve hydrostatic balance in a non-linear manner. Numerical experiments show that the new methods achieve high levels of consistency, maintaining hydrostatic and thermobaric equilibrium in the presence of strongly-sloping layer geometries, non-linear equations-of-state and non-uniform vertical stratification profiles. These results suggest that the new pressure gradient formulations may be appropriate for general circulation models that employ hybrid vertical coordinates and/or terrain-following representations.



## 1. Introduction

The development of flexible layered ocean models, capable of adapting to the complex vertical structure associated with stratified geophysical flows, represents an important ongoing numerical challenge in global climate modelling and numerical weather prediction. Compared to conventional *fixed-grid* formulations, layered models, in which the fluid is subdivided into a set of curvilinear layers, offer an opportunity to improve the fidelity with which vertical ocean transport processes are represented (Griffies et al., 2000). In this study, the issue of constructing a consistent and accurate numerical formulation for evaluation of the horizontal pressure gradient force in arbitrarily layered ocean models is discussed in detail. While seemingly innocuous, the development of stable and consistent discretisation schemes presents a significant numerical challenge, due to the complex interplay between non-linearities in the underlying fluid equation-of-state, the depth-wise stratification profiles, and the sloping geometry of the discrete fluid layers themselves.

This paper describes two new formulations for the pressure gradient operator that attempt to address these difficulties. We begin with a description of the overall numerical formulation, expressing the layered equations-of-motion in terms of an arbitrary vertical coordinate. In the following sections we briefly discuss several well-known instabilities associated with conventional horizontal pressure gradient formulations; review the semi-analytic approach of Adcroft et al. (2008); and then present our new techniques. Particular attention is paid to the development of flexible, high-order accurate numerical integration procedures, designed to preserve hydrostatic balance in the presence of the various non-linearities imposed by the thermodynamic and geometrical structure of the problem. The experimental results presented in Section 6 are designed to assess the consistency, accuracy and stability of the new schemes, contrasting the relative performance of the two new formulations for several two-dimensional *ocean-at-rest* type benchmark problems.

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http://dx.doi.org/10.1016/j.ocemod.2017.05.003

## 2. A simplified layered ocean model

Following Bleck (2002), Adcroft and Hallberg (2006), Higdon (2002); 2005) and Leclair and Madec (2011), the hydrostatic and non-Boussinesq equations of motion for a rotating geophysical fluid can be expressed in terms of a generalised vertical coordinate $s$ as a set of five prognostic conservation laws, with two equations for the horizontal velocity components, two balance-laws for a pair of thermodynamic variables, an evolution equation for a pressure *thickness* variable, and a diagnostic expression for the equation-of-state of the fluid. Specifically, adopting similar notion to that introduced in Bleck (2002), the continuous equations can be written as

$$\partial_t(\mathbf{u}_h) + (\mathbf{u}_h \cdot \nabla_s)\,\mathbf{u}_h + \dot{s}\,\partial_s(p)\,\partial_p(\mathbf{u}_h) + f\mathbf{u}_h^{\perp} = \nabla_s(\Phi) + \rho^{-1}\nabla_s(p) + \mathbf{F}_{\mathbf{u}_h}\,, \tag{1}$$

$$\partial_p(\Phi) = \rho^{-1}(T, S, p)\,, \tag{2}$$

$$\partial_t(\partial_s(p)) + \nabla_s \cdot (\mathbf{u}_h\,\partial_s(p)) + \partial_s(\dot{s}\,\partial_s(p)) = F_p\,, \tag{3}$$

$$\partial_t(\partial_s(p)\,T) + \nabla_s \cdot (\mathbf{u}_h\,\partial_s(p)\,T) + \partial_s(\dot{s}\,\partial_s(p)\,T) = F_T\,, \tag{4}$$

$$\partial_t(\partial_s(p)\,S) + \nabla_s \cdot (\mathbf{u}_h\,\partial_s(p)\,S) + \partial_s(\dot{s}\,\partial_s(p)\,S) = F_S\,. \tag{5}$$

Here $\mathbf{u}_h = (u, v)$ is the horizontal velocity field, $\mathbf{u}_h^{\perp} = (-v, u)$, and $f$ is the Coriolis parameter. $\Phi = gz$ is the geopotential, where $g$ is the acceleration due to gravity and $z$ is the height from a reference surface. The differential quantity $\partial_s(p)$ is a vertical *pressure-thickness* variable and $\dot{s}$ is an associated flow-rate, normal to surfaces of constant $s$. $T$ and $S$ are the scalar temperature and salinity distributions, respectively. Note that the specific choice of thermodynamic pairing is dependent on the equation of state used, with, for example, potential temperature and practical salinity $(T, S) = (\theta, S_p)$ used in a number of existing thermodynamic models (Wright, 1997), while recent formulations (Roquet et al., 2015; McDougall and Barker, 2011; Jackett et al., 2006) necessitate a switch to the conservative temperature and absolute salinity pair $(T, S) = (\Theta, S_A)$. The forcing terms $\mathbf{F}_{\mathbf{u}_h}$, $F_p$, $F_T$ and $F_S$ incorporate any additional sources and sinks associated with each quantity, in addition to the effect of generalised diffusion/mixing on both the momentum and thermodynamic variables, respectively. The fluid density $\rho = f(T, S, p)$ is diagnosed via a general non-linear equation of state, and the geopotential $\Phi = gz$ is expressed in terms of hydrostatic balance. The differential operator $\partial_t$ denotes a derivative with respect to time, $\partial_s$ denotes a derivative with respect to the generalised vertical coordinate $s$. $\nabla_s = (\partial_x, \partial_y, 0)$ is a layerwise gradient operator, taken along surfaces of constant $s$. Expressions for the transport of passive tracers can be added to this system via the inclusion of additional advection-diffusion equations of the form of (Eq. (5)).

In this study, a layered Arbitrary Lagrangian Eulerian (ALE) formulation is employed, discretising the vertical coordinate $s$ into a stack of discrete fluid layers, and setting the cross-coordinate flow-rate $\dot{s}$ to zero. Such a constraint implies dynamic motion of the layer interface surfaces themselves, with the thickness of the fluid layers evolving in time due to mass conservation. Integrating (Eq. (1)–(5)) over the vertical extent of each layer and setting $\dot{s} = 0$, the semi-discrete equations for a given layer $k$ can be written

$$\partial_t\big(\bar{\mathbf{u}}_{h,k}\big) + (\bar{\mathbf{u}}_h \cdot \nabla_s)\,\bar{\mathbf{u}}_h + f\bar{\mathbf{u}}_h^{\perp} = \frac{1}{\Delta p_k}\int_k \nabla_s(\Phi) + \rho^{-1}\nabla_s(p)\,\mathrm{d}p + \bar{\mathbf{F}}_{\mathbf{u}_h,k}\,, \tag{6}$$

$$\partial_t(\Delta p_k) + \nabla_s \cdot \big(\bar{\mathbf{u}}_{h,k}\,\Delta p_k\big) = \Delta p_k\,\bar{F}_{p,k}\,, \tag{7}$$

$$\partial_t\big(\Delta p_k\,\bar{T}_k\big) + \nabla_s \cdot \big(\bar{\mathbf{u}}_{h,k}\,\Delta p_k\,\bar{T}_k\big) = \Delta p_k\,\bar{F}_{T,k}\,, \tag{8}$$

$$\partial_t\big(\Delta p_k\,\bar{S}_k\big) + \nabla_s \cdot \big(\bar{\mathbf{u}}_{h,k}\,\Delta p_k\,\bar{S}_k\big) = \Delta p_k\,\bar{F}_{S,k}\,. \tag{9}$$

Here $(\bar{\cdot}) = \frac{1}{\Delta p}\int_{p_b}^{pt}(\,\cdot\,)\,\mathrm{d}p$ denotes a layer-mean quantity, integrated between the lower and upper interfaces $p_b$ and $p_t$ that define the vertical extent of each layer with respect to pressure. The associated discrete *pressure-thickness* variable $\Delta p$ is simply the difference in layer interface pressures $\Delta p = p_b - p_t$. Integrals of the pressure gradient terms in the momentum equation (Eq. (6)) have be written explicitly here for consistency with the finite-volume type formulations developed in Sections 4 and 5.

### 2.1. Existing formulations for the horizontal pressure gradient operator

Numerical issues related to the discretisation of the horizontal pressure gradient force have long plagued the development of layered ocean models. These numerical errors typically manifest as spurious horizontal accelerations, causing the model to erroneously 'drift' away from the desired equilibrium state over time. The genesis of such difficulties can be explained by examining the interaction of the two differential operators associated with the pressure gradient force in Eq. (6)

$$\mathrm{PGF} = \nabla_s(\Phi) + \rho^{-1}\nabla_s(p)\,. \tag{10}$$

Given particular (conventional) choices of vertical coordinate, namely $s = z$ or $s = p$, the form of the pressure gradient operator can be simplified, with one of the two gradient terms ($\nabla_s(\Phi)$ and $\rho^{-1}\nabla_s(p)$) evaluating to zero. Specifically, in conventional height-based coordinates $\nabla_s(\Phi) = \nabla_z(\Phi) = 0$, while in a pressure-based coordinate system $\rho^{-1}\nabla_s(p) = \rho^{-1}\nabla_p(p) = 0$. Unfortunately, this exact cancellation is not preserved when adopting arbitrary vertical coordinate systems appropriate for layered ocean modelling, such as terrain-following coordinates and/or time- and space-dependent Lagrangian representations. In such cases, a straightforward discretisation of the two gradient operators in Eq. (10) can lead to inconsistencies, with the interaction of the numerical truncation errors associated with each gradient term leading to inexact cancellation. Noting that the magnitude of these two terms is typically large compared to the dynamical signal (Adcroft et al., 2008), it can be understood that residual errors in the evaluation of the pressure gradient force can lead to non-negligible spurious horizontal motion. This behaviour is exacerbated when the fluid layers are steeply sloping and the imposed thermodynamic stratification profiles are highly non-uniform.

Conventionally, layered isopycnic-type models (Bleck, 2002) have sought to exploit the so-called *Montgomery-potential* form of the horizontal pressure gradient operator. Setting $M = p/\rho + \Phi$, the horizontal acceleration can be transformed as follows

$$\mathrm{PGF} = \nabla_s(M) + p\nabla_s(\rho^{-1})\,. \tag{11}$$

Note that in an exact density-following coordinate system ($s = \rho$), the second term in Eq. (11) can be seen to vanish, with $p\nabla_s(\rho^{-1}) = p\nabla_\rho(\rho^{-1}) = 0$. While such a result is attractive from a theoretical standpoint, it should be noted that practical isopycnic-type models do not typically adopt a coordinate system based on the exact in-situ densities, preferring instead hybrid potential-density-based representations, with height-based transitions employed near layer outcropping (Bleck, 2002). Nonetheless, it can be argued that use of the Montgomery potential form serves to mitigate associated numerical errors, through a minimisation of the