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Revisiting floating bodies

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Abstract

The conic sections, as well as the solids obtained by revolving these curves, and many of their surprising properties, were already studied by Greek mathematicians since at least the fourth century B.C. Some of these properties come to the light, or are rediscovered, from time to time. In this paper we characterize the conic sections as the plane curves whose tangent lines cut off from a certain similar curve segments of constant area. We also characterize some quadrics as the surfaces whose tangent planes cut off from a certain similar surface compact sets of constant volume. Our work is developed in the most general multidimensional case.

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1. Introduction

Throughout literature, we can find several properties of the plane regions obtained by cutting off with a line from conic sections and of the solids obtained by cutting off with a plane from quadrics of revolution. Let us show a brief chronological exposition of some results concerning the topic so far.

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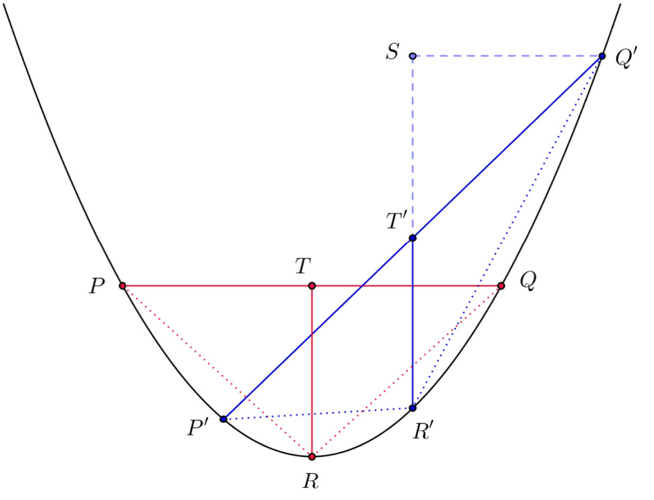

Fig. 1. T and T' are midpoints of the respective chords. When $TR = T'R'$, then $QT = Q'S$, and the parabolic segments cut off by the chords PQ and $P'Q'$ have equal area.

In his book *Quadrature of the Parabola* (QP), Archimedes proved the following result (see [13, (QP), Propositions 17 and 24, p. 246–252]):

Proposition 1 (*QP*)**.** *The area of any segment of a parabola is four thirds of the area of the triangle which has the same base as the segment and equal height.*

Here, the *base* of the parabolic segment is just the chord of the segment. The *height* is the segment TR, where T is the midpoint of the base and R is the point where the line parallel to the axis of the parabola through T meets the parabola, see Fig. 1. The line tangent to the parabola in R is parallel to the base. The point R is the *vertex* of the segment [13, (QP), Proposition 1, p. 234 and Proposition 18, p. 247].

Moreover, Archimedes uses the above result at the beginning of his book *On Conoids and Spheroids* (CS) to prove (see [20, (CS), Proposition III; t. I, p. 149–152]), see Fig. 1:

Proposition 2 (*CS*)**.** *If from a same parabola two segments are cut off in any way, and the diameters of the segments are equal, then the segments will have equal area.*

Here, Archimedes calls *diameter* the segment TR that he called *height* in (QP). In fact, TR bisects all the chords parallel to the base of the segment.

The first propositions in (CS) concern plane geometry, some of them dealing with the quadrature of the ellipse (see [13, (CS), Propositions 4, 5, and 6, p. 113–115]) (we do not know any result by Archimedes related to the quadrature of elliptic or hyperbolic segments). But the main target in (CS) is to show the volumes of the solid segments cut off by a plane from a paraboloid of revolution (or *right-angled conoid*), a two-sheeted hyperboloid of revolution (or *obtuse-angled conoid*), and an ellipsoid of revolution (or *spheroid*).

In this way, generalizing the main result in (QP), Archimedes proves in (CS) (see [13, (CS), Propositions 21 and 22, p. 131–133]):

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