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Witt's cancellation theorem seen as a cancellation

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Received 16 July 2016; received in revised form 11 September 2016

Dedicated to the late Professor Amit Roy

Abstract

The year 2017 marks the 80th anniversary of Witt's famous paper containing key results, including the Witt cancellation theorem, which form the foundation for the algebraic theory of quadratic forms. We pay homage to this paper by presenting a transparent and algebraic proof of the Witt cancellation theorem, which itself is based on a cancellation. We also present an overview of some recent spectacular work which is still building on Witt's original creation of the algebraic theory of quadratic forms.

MSC 2010: primary 11E81; secondary 19D45

keywords: Quadratic forms; Witt rings; Milnor K-theory

1. Introduction

The algebraic theory of quadratic forms will soon celebrate its 80th birthday. Indeed, it was 1937 when Witt's pioneering paper [24] – a mere 14 pages – first introduced many beautiful results which form the foundation for the algebraic theory of quadratic forms.

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http://dx.doi.org/10.1016/j.exmath.2016.10.003

In particular, Witt describes the construction of the Witt ring itself and the essential fact needed for its construction over an arbitrary field. This result, originally Satz 4 in [24], is now formulated as the Witt cancellation theorem, and it is the technical heart of Witt's brilliant idea to study the collection of *all* quadratic forms over a given field as a single algebraic entity. Prior to Witt's paper, quadratic forms were studied one at a time. However Witt showed that a certain collection of quadratic forms under an equivalence relation can be equipped with the structure of a commutative ring. Indeed, Satz 6 says:

"*Die Klassen ähnlicher Formen bilden einen Ring*"

which means, "The classes of similar forms, form a ring". In order to honor Witt's contributions, this ring is now called the *Witt ring*.

The Witt ring remains a central object of study, even 80 years after its birth. Building on Voevodsky's Fields medal winning work from 2002, Orlov, Vishik and Voevodsky settled Milnor's conjecture [12] on quadratic forms, which is a deep statement about the structure of the Witt ring. This work uses sophisticated tools from algebraic geometry and homotopy theory to provide a complete set of invariants for quadratic forms, extending the classical invariants known to Witt [24], including dimension, discriminant and the Clifford invariant.

In addition to its crucial role in defining the Witt ring, the Witt cancellation theorem also has other important applications, such as establishing Sylvester's law of Inertia, which classifies quadratic forms over the field of real numbers. Clearly the Witt cancellation theorem is special and therefore deserves further analysis. The main goal of this paper is to present a transparent and algebraic proof to complement the classical geometric proof, and then carefully compare the two approaches.

The paper is organized as follows. In Section 2 we state the Witt cancellation theorem, guide the reader towards our proof of the cancellation theorem, and then present the proof itself. A geometric approach to Witt cancellation, based on hyperplane reflections, is presented in Section 3. In Section 4 we provide a comparison between the algebraic and geometric approaches. Using Witt cancellation as the key, we review the construction of the Witt ring of quadratic forms in Section 5. In Section 6 we present an informal overview of the Milnor conjectures on quadratic forms and some dramatic recent developments culminating in the proof of the Bloch–Kato conjecture, which is a considerable extension of part of the original Milnor conjectures. In Section 7 we reveal an interesting surprise. All sections, except possibly Section 6, can be read profitably by any reader who is familiar with basic linear algebra.

We begin with some preliminaries. Throughout the paper we assume that our base field F has characteristic not equal to 2. There are several equivalent definitions of a quadratic form. The following is probably the most commonly used definition. An n-ary *quadratic form* q over F is a homogeneous polynomial of degree 2 in n variable over F:

$$q = \sum_{i,j=1}^{n} a_{ij} x_i x_j \quad \text{for } a_{ij} \text{ in } F.$$

It is customary to render the coefficients symmetric by writing

$$q = \sum_{i,j=1}^{n} b_{ij} x_i x_j, \quad \text{where } b_{ij} = \frac{a_{ij} + a_{ji}}{2},$$

Please cite this article in press as: S.K. Chebolu, et al., Witt's cancellation theorem seen as a cancellation, Expo. Math. (2016), http://dx.doi.org/10.1016/j.exmath.2016.10.003