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Representability in supergeometry

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Abstract

In this paper we use the notion of Grothendieck topology to present a unified way to approach representability in supergeometry, which applies to both the differential and algebraic settings.

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1. Introduction

Supergeometry is the mathematical tool originally developed to study supersymmetry. It was discovered, in the early 1970s, by the physicists Wess and Zumino [18] and Salam and Strathdee [15] among others. Supergeometry grew out of the works of Berezin [5], Kostant [12] and Leites [13], then, later on, by Manin [14], Bernstein [8] and others. These authors introduced an algebraic point of view on differential geometry, with emphasis on the methods that were originally developed in algebraic geometry by Grothendieck to handle schemes. In particular, the functor of points approach invented by Grothendieck turned out

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to be very useful to formalize the physical “anticommuting variables” of supersymmetry and provided a very useful tool to link algebra and geometry in a categorical way. The language developed by Grothendieck is, in fact, powerful enough to reveal the geometric nature not just of superschemes, but also of supermanifolds and superspaces in general.

In this paper we want to examine representability in the supergeometric context, with the use of Grothendieck topologies. In particular, we are able to prove a representability criterion (see Theorem 4.8), that can be applied to functors $\mathcal{C}^{\mathrm{op}} \longrightarrow (\mathrm{Set})$, where $\mathcal{C}$ is a *superspace site*, that is a full subcategory of the category of superspaces (SSpaces), with some additional very natural properties (see Definition 4.1). This broadens the range of application of the criterion, first published in [6], to include, not only superschemes or supermanifolds, but also some less trivial categories like Leites regular supermanifolds and locally finitely generated superspaces, introduced by Alldridge et al. in [1]. Our hope is that more general objects can be studied using this criterion, which formalizes the ideas of Grothendieck, adapting them to the supergeometric context.

2. Preliminaries on Grothendieck topologies

We start with the notion of *Grothendieck topology*. For more details we refer the reader to [2], Exposé ii,[1] [17,7].

Definition 2.1. Let $\mathcal{C}$ be a category. A *Grothendieck topology* $\mathcal{T}$ on $\mathcal{C}$ assigns to each object $U \in \mathrm{Ob}(\mathcal{C})$ a collection $\mathrm{Cov}(U)$ whose elements are families of morphisms with fixed target U, with the following properties.

(1) If $V \to U$ is an isomorphism, $\{V \to U\} \in \mathrm{Cov}(U)$.
(2) If $V \to U$ is an arrow and $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(U)$, the fibered products $\{U_i \times_U V\}$ exist and the collection of projections $\{U_i \times_U V \to V\}_{i\in I} \in \mathrm{Cov}(V)$.
(3) If $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(U)$ and for each i we have that $\{V_{ij} \to U_i\}_{j\in J_i} \in \mathrm{Cov}(U_i)$, then $\{V_{ij} \to U_i \to U\}_{i\in I, j\in J_i} \in \mathrm{Cov}(U)$.

The pair $(\mathcal{C}, \mathcal{T})$ is called a *site*. The elements of $\mathrm{Cov}(U)$ are called *coverings*.

We may abuse the notation and write $\mathcal{U} \in \mathcal{T}$ or $\mathcal{U} \in \mathrm{Cov}(\mathcal{C})$ to indicate that $\mathcal{U} = \{U_i \to U\}_{i\in I}$ is a covering for the topology $\mathcal{T}$.

Now we discuss some key examples, which will be fundamental for our treatment.

Example 2.2. 1. Let us consider a topological space X and set $\mathcal{X}_{cl}$ to be the category with open sets as objects and inclusions as arrows. We say $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(U)$ if and only if $\bigcup_{i\in I} U_i = U$. We obtain a site $(X_{cl}, \mathcal{X}_{cl})$.

2. Let us consider the category (Sch) of schemes and define coverings of U to be collections of open embeddings whose images cover U. This is a topology, because of the existence and the properties of the fibered product in (Sch). This is called the *Zariski topology*.

Now we want to compare different topologies on the same category.

[1] In [2] what we call “Grothendieck topology” is called “pretopology”, we adhere to the terminology in [17].

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