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A note on the equioscillation theorem for best ridge function approximation

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Abstract

We consider the approximation of a continuous function, defined on a compact set of the d-dimensional Euclidean space, by sums of two ridge functions. We obtain a necessary and sufficient condition for such a sum to be a best approximation. The result resembles the classical Chebyshev equioscillation theorem for polynomial approximation.
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1. Introduction

Let Q be compact set in the d-dimensional Euclidean space and $C(Q)$ be the space of continuous real-valued functions on Q. Consider the approximation of a function $f \in C(Q)$ by sums of the form $g_1(\mathbf{a}_1 \cdot \mathbf{x}) + g_2(\mathbf{a}_2 \cdot \mathbf{x})$, where $\mathbf{a}_i$ are fixed vectors (directions) in $\mathbb{R}^d \setminus \{\mathbf{0}\}$ and g_i are continuous univariate functions. We are interested in characterization of a best approximation. Note that functions of the form $g(\mathbf{a} \cdot \mathbf{x})$ are

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called ridge functions. These functions and their linear combinations arise naturally in problems of computerized tomography (see, e.g., [26]), statistics (see, e.g., [9,11]), partial differential equations [19] (where they are called *plane waves*), neural networks (see, e.g., [33] and references therein), and approximation theory (see, e.g., [17,27,31,32]). In the past few years, problems of ridge function representation have gained special attention among researchers (see e.g. [1,24,25,34]). For more on ridge functions and application areas see a recently published monograph by Pinkus [35].

Characterization theorems for best approximating elements are essential in approximation theory. The classical and most striking example of such a theorem are the Chebyshev equioscillation theorem. This theorem characterizes the unique best uniform approximation to a continuous real valued function $F(t)$ by polynomials $P(t)$ of degree at most n, by the oscillating nature of the difference $F(t) - P(t)$. The result says that if such polynomial has the property that for some particular $n + 2$ points t_i in [0, 1]

$$F(t_i) - P(t_i) = (-1)^i \max_{x\in[0,1]} |F(t) - P(t)|, \quad i = 1, \ldots, n+2,$$

then P is the best approximation to F on [0, 1]. The monograph of Natanson [30] contains a very rich commentary on this theorem. Some general alternation type theorems applying to any finite dimensional subspace E of $C(I)$ for I a cell in $\mathbb{R}^d$, may be found in Buck [5]. For a short history and various modifications of the Chebyshev alternation theorem see [4].

In this note, we obtain an equioscillation theorem for approximation of multivariate functions by sums of two ridge functions. To be more precise, let Q be a compact subset of the space $\mathbb{R}^d$. Fix two directions $\mathbf{a}_1$ and $\mathbf{a}_2$ in $\mathbb{R}^d$ and consider the following space

$$\mathcal{R} = \mathcal{R}(\mathbf{a}_1, \mathbf{a}_2) = \{g_1(\mathbf{a}_1 \cdot \mathbf{x}) + g_2(\mathbf{a}_2 \cdot \mathbf{x}) : \ g_1, g_2 \in C(\mathbb{R})\}.$$

Note that the space $\mathcal{R}$ is a linear space. Assume a function $f \in C(Q)$ is given. We ask and answer the following question: which geometrical conditions imposed on $G_0 \in \mathcal{R}$ is necessary and sufficient for the equality

$$\|f - G_0\| = \inf_{G\in\mathcal{R}} \|f - G\| \, ? \tag{1.1}$$

Here $\|\cdot\|$ denotes the standard uniform norm in $C(Q)$. Recall that functions G_0 satisfying (1.1) are called best approximations or extremal elements.

It should be remarked that in the special case when $Q \subset \mathbb{R}^2$ and $\mathbf{a}_1$ and $\mathbf{a}_2$ coincide with the coordinate directions, the above question was answered by Khavinson [20]. In [20], he obtained an equioscillation theorem for a best approximating sum $\varphi(x) + \psi(y)$. In our papers [14,17], Chebyshev type theorems were proven for ridge functions under additional assumption that Q is convex. For a more recent and detailed discussion of an equioscillation theorem in ridge function approximation see Pinkus [35].

2. Equioscillation theorem for ridge functions

We start with a definition of paths with respect to two directions. These objects will play an essential role in our further analysis.

Definition 2.1 (*See [14]*)**.** A finite or infinite ordered set $p = (\mathbf{p}_1, \mathbf{p}_2, \ldots) \subset Q$ with $\mathbf{p}_i \neq \mathbf{p}_{i+1}$, and either $\mathbf{a}_1 \cdot \mathbf{p}_1 = \mathbf{a}_1 \cdot \mathbf{p}_2, \mathbf{a}_2 \cdot \mathbf{p}_2 = \mathbf{a}_2 \cdot \mathbf{p}_3, \mathbf{a}_1 \cdot \mathbf{p}_3 = \mathbf{a}_1 \cdot \mathbf{p}_4, \ldots$ or

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