The Cardinal Krull Dimension of a Ring of Holomorphic Functions

Pete L. Clark

Department of Mathematics, Boyd Graduate Studies Research Center, University of Georgia, Athens, GA 30602 7403, USA
plclark@gmail.com

Abstract

We give an exposition of a recent result of M. Kapovich on the cardinal Krull dimension of the ring of holomorphic functions on a connected $\mathbb{C}$-manifold. By reducing to the one-dimensional case we give a stronger lower bound for Stein manifolds.

1. Introduction

Recently Kapovich proved the following striking result: if a connected $\mathbb{C}$-manifold M has a noncontant holomorphic function, then the ring $\mathrm{Hol}(M)$ of holomorphic functions on M has a chain of prime ideals of continuum length [9].

Here we give an exposition of Kapovich's Theorem, in which the algebraic part of the proof is shortened and simplified. Kapovich defines a class of "ample rings" and shows – by a beautiful use of Sard's Theorem that we follow closely – that if $\mathrm{Hol}(M) \supsetneq \mathbb{C}$ then $\mathrm{Hol}(M)$ is ample, and then he shows that any ample ring has a chain of prime ideals of continuum length [9, Thm. 4]. The proofs, and even the definition of ample rings, make use of hyperreals and hypernatural numbers.

We observe that the analytic part of Kapovich's argument shows that if $\mathrm{Hol}(M) \supsetneq \mathbb{C}$ then $\mathrm{Hol}(M)$ admits an infinite sequence of discrete valuations $\{v_k\}$ which are independent in the sense that for any sequence $\{n_k\}$ of natural numbers there is $f \in \mathrm{Hol}(M)$ such that $v_k(f) = n_k$ for all k, and then we show (Theorem 1.4) that any ring admitting a sequence of independent discrete valuations has a chain of prime ideals of continuum length. In place of nonstandard analysis our proofs use *ultralimits*. In order to understand these, a reader need only know what an ultrafilter is and that an ultrafilter on a compact space converges to a unique point.

Although $\mathfrak{c}$ is a large number, it may not be large enough! Results of Henriksen [7] and Alling [1], [2] show that if M is a noncompact Riemann surface then $\mathrm{Spec}\,\mathrm{Hol}(M)$ has a chain of length $2^{\aleph_1}$. Building on these results we show (Theorem 3.3) that for a connected $\mathbb{C}$-manifold M, if $M \cong V \times N$ for a Stein manifold V of positive dimension, then $\mathrm{Hol}(M)$ admits a chain of prime ideals of length $2^{\aleph_1}$.