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Orthogonality of compact operators

Paweł Wójcik

Institute of Mathematics, Pedagogical University of Cracow, Podchorążych 2, 30-084 Kraków, Poland

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Abstract

In this paper we characterize the Birkhoff–James orthogonality for elements of $\mathcal{K}(X;Y)$. In this way we extend the Bhatia–Šemrl theorem. As an application, we consider the approximate orthogonality preserving property. Moreover, we give a new characterization of inner product spaces.

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1. Introduction

We start with some notation which will be of use later. Let $(X, \|\cdot\|)$ be a normed space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. By $S(X)$ we denote the unit sphere in a normed space X. If the norm comes from an inner product $\langle\cdot|\cdot\rangle$, there is one natural orthogonality relation: $x \perp y :\Leftrightarrow \langle x|y\rangle = 0$. In general case, there are several notions of orthogonality and one of the most outstanding is the definition introduced by Birkhoff [4] (cf. also James [11]). For $x, y \in X$ we define:

$$x \perp_B y \quad :\Leftrightarrow \quad \forall_{\lambda \in \mathbb{K}}: \quad \|x\| \leqslant \|x + \lambda y\|.$$

E-mail address: pwojcik@up.krakow.pl.

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This relation is clearly homogeneous, but neither symmetric nor additive, unless the norm comes from an inner product. Of course, in an inner product space we have $\perp_B = \perp$.

The dual space is denoted by X^*. It is easy to see that for two elements x, y of a normed linear space X, it holds $x \perp_B y$ if and only if there is a norm one linear functional $f \in X^*$ such that $f(x) = \|x\|$ and $f(y) = 0$. If we have additional structures on a normed linear space X, then we get other characterizations of the Birkhoff orthogonality. One of the first results of this form is the result obtained by Bhatia and Šemrl [2] for the Banach space $\mathcal{L}(\mathcal{H})$ of all bounded linear operators on a Hilbert space $\mathcal{H}$.

Theorem 1.1 ([2]). *Let $\mathcal{H}$ be a complex Hilbert space. Let $A, B \in \mathcal{L}(\mathcal{H})$. If $\dim \mathcal{H} < \infty$, then $A \perp_B B$ if and only if there is a unit vector $x \in \mathcal{H}$ such that $\|Ax\| = \|A\|$ and $\langle Ax|Bx\rangle = 0$.*

In particular, Bhatia and Šemrl [2] proved that if X is a real or complex finite-dimensional inner product space and $A, B \in \mathcal{L}(\mathcal{H})$ then

$$A \perp_B B \quad \Leftrightarrow \quad \exists_{u \in S(X)}\ \|Au\| = \|A\|, \quad Au \perp_B Bu. \tag{1.1}$$

In the paper [2] it is conjectured that (1.1) is valid for any finite-dimensional normed space X. Li and Schneider [14] give a counterexample to the above conjecture. They show that it does not hold for the space $X = l_p^n$, with $p \neq 2$. Benítez, Fernández and Soriano [1] extended this result. Namely, they proved the following theorem.

Theorem 1.2. *A real finite-dimensional normed space X is an inner product space if and only if, for all $A, B \in \mathcal{L}(X)$ we have*

$$A \perp_B B \quad \Leftrightarrow \quad \exists_{u \in S(X)}\ \|Au\| = \|A\|, \quad Au \perp_B Bu.$$

Remark 1. Bhattacharyya and Grover [3] gave another proof of the Bhatia–Šemrl theorem using tools of convex analysis. Namely, they considered a convex function $\varphi(t) := \|A + tB\|$ and its subdifferential.

Let $\mathcal{L}(X; Y)$ be the space of all linear, continuous operators from X into Y. In this paper, we prove similar criteria in the case $\mathcal{K}(X; Y)$, where $\mathcal{K}(X; Y)$ denotes the space of all compact operators going from a normed space X to a Banach space Y. In particular, we generalize Theorem 1.1.

2. Preliminaries

By $S(X)$ we denote the unit sphere in a normed space X and by $\mathrm{Ext}(S(X))$ the set of its extreme points. Even if the norm in X does not come from an inner product there always exists (as noticed by G. Lumer [15] and J.R. Giles [10], cf. also [9]) a mapping $[\cdot|\cdot] : X \times X \to \mathbb{K}$ satisfying the properties:

(sip1) $\forall_{x,y,z \in X}\ \forall_{\alpha,\beta \in \mathbb{K}} :\ [\alpha x + \beta y|z] = \alpha\,[x|z] + \beta\,[y|z]$;
(sip2) $\forall_{x,y \in X}\ \forall_{\alpha \in \mathbb{K}} :\ [x|\alpha y] = \overline{\alpha}\,[x|y]$;
(sip3) $\forall_{x,y \in X} :\ |\,[x|y]\,| \leqslant \|x\| \cdot \|y\|$;
(sip4) $\forall_{x \in X} :\ [x|x] = \|x\|^2$.