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# Spectral sequences, exact couples and persistent homology of filtrations

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**Abstract**

In this paper we study the relationship between a very classical algebraic object associated to a filtration of topological spaces, namely a spectral sequence introduced by Leray in the 1940s, and a more recently invented object that has found many applications—namely, its persistent homology groups. We show the existence of a long exact sequence of groups linking these two objects and using it derive formulas expressing the dimensions of each individual groups of one object in terms of the dimensions of the groups in the other object. The main tool used to mediate between these objects is the notion of exact couples first introduced by Massey in 1952.


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***Keywords:*** Spectral sequence; Filtration; Persistent homology; Exact couple

## 1. Introduction

Given a topological space $X$ (which for the purposes of the current paper will be taken to be a finite CW-complex) a finite filtration, $\mathcal{F}$ of $X$, is a sequence of subspaces

$$\emptyset = X_{-1} = X_0 \subset X_1 \subset \cdots \subset X_N = X_{N+1} = \cdots = X$$

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(we will denote the subspace $X_p$ in the sequence by $\mathcal{F}_{-}X$). A very classical technique in algebraic topology for computing topological invariants of a space $X$ is to consider a filtration $\mathcal{F}$ of $X$ where the successive spaces $\mathcal{F}_s X$ capture progressively more and more of the topology of $X$. For example, in case $X$ is a CW-complex one can take for $\mathcal{F}_p X$ the $p$th skeleton $\mathrm{sk}_p(X)$ consisting of all cells of dimension at most $p$. More generally, given a cellular map $f : X \to Y$, one can take for $\mathcal{F}_p X$ the inverse image under $f$ of $\mathrm{sk}_p(Y)$. One then associates to this sequence a sequence of algebraic objects which in nice situations is expected to "converge" (in an appropriate sense) to the topological invariant (such as the homology or cohomology groups) associated to $X$ itself, directly computing which is often an intractable problem. This sequence of algebraic approximations is called a *spectral sequence* associated to the filtration $\mathcal{F}$, and was first introduced by Leray [11] in 1946 (see also the book by Dieudonné [6, page 137] for a comprehensive historical survey).

Spectral sequences are now ubiquitous in mathematics. A typical application which is common in discrete geometry, as well as in quantitative real algebraic geometry, is to use the initial terms of a certain spectral sequence to give upper bounds on the topological complexity (for example, the sum of Betti numbers) of the object of interest $X$ (often a semi-algebraic subset of some $\mathbb{R}^n$) (see for example, [2,9] for applications of this kind). Spectral sequences also have algorithmic applications in the context of computational geometry (see for example [1]).

Much more recently the notion of *persistent homology* [7,16] associated to a filtration has become an important tool in various applications. In contrast to spectral sequences discussed in the previous paragraph, the emphasis here is not so much on studying the topology of the final object $X$, but rather on the intermediate spaces of the filtration. Indeed the final object $X$ in many cases is either contractible or homologically trivial. For example, this is the case for filtrations arising from alpha-complexes introduced by Edelsbrunner et al. in [8]. The *persistent homology groups* (see Definition 8 for a precise definition) are defined such that their dimensions equal the dimensions of spaces of homological cycles that appear at a certain fixed point of the filtration $\mathcal{F}$ and disappear at a certain (later) point. The homological cycles that persist for long intervals often carry important information about the underlying data sets that give rise to the filtration, and this is why computing them is important in practice. We refer the reader to survey articles [5,10,16] for details regarding these applications.

While spectral sequences and persistent homology were invented for entirely different purposes as explained above, they are both associated to filtrations of topological spaces and it is natural to wonder about the exact relationship between these two notions, and in particular, whether the dimensions of the groups appearing in the spectral sequence of a filtration carries any more information than the dimensions of the persistent homology groups of the same filtration. One of the results in this paper (Theorem 1) shows that this is not the case, and the dimensions of the groups appearing in the spectral sequence of a filtration can be recovered from the dimensions of its persistent homology groups. It has been observed by several authors (see for example, [5,7,10]) that there exists a close connection between the spectral sequence of a filtration and its persistent homology groups. The goal of this paper is to make precise this relationship—in particular, to derive formulas which expresses the dimensions of each group appearing in the spectral sequence of a filtration in terms of the persistent Betti numbers and vice versa.

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