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Hypersurfaces achieving the Homma–Kim bound ☆

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ABSTRACT

Let X^n be a hypersurface in $\mathbb{P}^{n+1}$ with $n \geq 2$ defined over a finite field. The main result of this note is the classification, up to projective equivalence, of hypersurfaces X^n without a linear component when the number of their rational points achieves the Homma–Kim bound.

1. Introduction

In a series of papers [4–6], Homma and Kim settled the Sziklai conjecture [11] for plane curves. In particular, as a consequence of their results one can deduce that for any plane curve C of degree d over a finite field $\mathbb{F}_q$ of q elements without $\mathbb{F}_q$-linear components,

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the number $N_q(C)$ of $\mathbb{F}_q$-points of C is bounded by $N_q(C) \leq (d-1)q+2$ and equality holds if and only if $d = q = 4$ and C is projectively equivalent to the plane curve

$$X_0^4 + X_1^4 + X_2^4 + X_0^2X_1^2 + X_1^2X_2^2 + X_0^2X_2^2 + X_0^2X_1X_2 + X_0X_1^2X_2 + X_0X_1X_2^2 = 0.$$

In [7] the authors establish also an upper bound for the number $N_q(X^n)$ of $\mathbb{F}_q$-points of a hypersurface $X^n \subset \mathbb{P}^{n+1}$ of degree d and dimension $n \geq 2$ defined over $\mathbb{F}_q$ and without an $\mathbb{F}_q$-linear component which is an analogous to their bound for a plane curve. Moreover, they show that their upper bound

$$\Theta_n^{d,q} := (d-1)q^n + dq^{n-1} + q^{n-2} + \cdots + q + 1$$

is the best one for irreducible hypersurfaces that is linear on their degrees, because, for each finite field, they give three nonsingular surfaces of different degrees that reach their bound.

In line with the above results for plane curves we characterize, up to projective equivalence, all the hypersurfaces $X^n \subset \mathbb{P}^{n+1}$ defined over $\mathbb{F}_q$ and without $\mathbb{F}_q$-linear components which reach the Homma–Kim bound $\Theta_n^{d,q}$ by proving the following classification result.

Theorem 1. *Let $X^n \subset \mathbb{P}^{n+1}$ be a hypersurface of degree $d \geq 2$ and dimension $n \geq 2$ defined over $\mathbb{F}_q$ and without $\mathbb{F}_q$-linear components. Then $N_q(X^n) \leq \Theta_n^{d,q}$ and equality holds if and only if $d \leq q+1$ and one of the following possibilities occurs:*

(1) *$d = q+1$ and X^n is a space-filling hypersurface*

$$(X_0, \ldots, X_{n+1})\ A\ {}^t(X_0^q, \ldots, X_{n+1}^q) = 0,$$

where $A = (a_{ij})_{i,j=1,\ldots,n+2}$ is an $(n+2) \times (n+2)$ matrix such that ${}^tA = -A$ and $a_{kk} = 0$ for every $k = 1, \ldots, n+2$; moreover, X^n is nonsingular if and only if $\det A \neq 0$;

(2) *$d = \sqrt{q}+1$ and X^n is projectively equivalent to a cone over the nonsingular Hermitian surface*

$$X_0^{\sqrt{q}+1} + X_1^{\sqrt{q}+1} + X_2^{\sqrt{q}+1} + X_3^{\sqrt{q}+1} = 0;$$

(3) *$d = 2$ and X^n is projectively equivalent to a cone over the hyperbolic quadric surface*

$$X_0X_2 + X_1X_3 = 0.$$

Finally, in the nonsingular case, we obtain the following immediate consequence of Theorem 1.